Mathematical Reviews

Vol. 18, No. 10 NOVEMBER 1957 pp. 785-862

FOUNDATIONS, THEORY OF SETS, LOGIC

Skolem, Th. An ordered set of arithmetic functions representing the least ε-number. Norske Vid. Selsk. Forh., Trondheim 29 (1956), 54–59 (1957).

Let S denote the smallest set of functions that satisfy (1), (2), and (3) below. (1) 0 and 1 are in S. (2) If $f(x)$ and $g(x)$ are in S, then so is $f(x)+g(x)$. (3) If $f(x)$ is in S, then so is $x^{f(x)}$ (While not stated explicitly, it is implied that x ranges over the non-negative integers). Write $f<g$ if $f(x)<g(x)$ for all $x>$ a certain x. Then S is simply ordered and its order type is the first ε-number. *S. Ginsburg.*

Łoś, Jerzy. The algebraic treatment of the methodology of elementary deductive systems. Studia Logica 2 (1955), 151–212. (Polish and Russian summaries)

Originally this paper was intended to be an exposition of Gödel's completeness theorem, but the later parts do contain some related new results and new proofs of known results.

Starting from some predicates r_i of n_i variables and functions f_j of k_j variables, the truth functional connectives $\rightarrow$, $'$ and quantifiers, let S be the set of all wffs (open and closed) which are obtained from these by composition in the usual manner, O the corresponding set of open formulae (containing no bound variables). For any operation of consequence Cn, a set X of formulas is called a system if $\mathrm{Cn}(X)=X$, X is called consistent if $\mathrm{Cn}(X)\neq S$, X is called complete if $\mathrm{Cn}(X)\neq 0$ and for any $\alpha\notin S$: $\mathrm{Cn}(X+\{\alpha\})=S$. We shall here consider the following rules of consequence: (a) one of the sets of axiom schemata for the propositional calculus, (b) the rule of substitution for individual variables, (c) the rule of modus ponens, (d) the rules of quantification. The operations of consequence Cn_S, Cn_O, Cn_I are now defined: Given a set $X\subseteq S$, $\mathrm{Cn}_S(X)$ will be the set of formulas which can be deduced from S by (a), (b), (c), (d); $\mathrm{Cn}_O(X)$ the set deduced by (a), (b), (c); and $\mathrm{Cn}_I(X)$ the set deduced by (a), (c). Much of the remainder of the paper depends on Lindenbaum's theorem, proved using Zorn's Lemma: If X_1 is a consistent system, then there is a complete system X_0 which includes X_1.

By a model is meant a sequence

$$\mathfrak{M}=\langle A, R_1, R_2, \cdots, F_1, F_2, \cdots\rangle,$$

where A is a non-empty set, $R\subseteq A_i^{n_i}$ for some integer n_i, and F_j is a function from $A_j^{k_j}$ into A, for some integer k_j. The set of all O-formulas valid in $\mathfrak{M}$ will be denoted by $O(\mathfrak{M})$, the set of all S-formulas valid in $\mathfrak{M}$ by $S(\mathfrak{M})$. If X is a system which is complete with respect to the operation I and Θ is the set of terms of X and $r_1, r_2, \cdots$ are the atomic relations of X, and $f_0, f_1, \cdots$ the functions of X, we define relation R_i^X such that R_i^X holds between the terms $\theta_1, \cdots, \theta_{n_i}$ if and only if the atomic formula $r_i(\theta_1, \cdots, \theta_2)$ belongs to X. Thus for any I-complete system X we can define a model

$$\mathfrak{L}^X=\langle\Theta, R_1^X, R_2^X, \cdots, \varphi_1, \varphi_2, \cdots\rangle,$$

where $\varphi_1, \varphi_2, \cdots$ are functions on terms with terms as values and correspond to the functions $f_1, f_2, \cdots$ of X. A formula α of an I-system X is said to belong to the kernel $K(X)$ of X if α and all its substitutes by rule (b) belong to X. These notions allow rather neat formulations and proofs of many classical theorems and some new ones. Among the most important of the theorems proved here are the following. 1) $O(\mathfrak{L}^X)=K(X)$. 2) Every I-consistent system is the common part of all I-complete systems containing it: If X is an I-system, then $X=\prod X_i$ ($X\subseteq X_i$, X_i I-complete). 3) All true formulae of the propositional calculus and only such formulae follow from the axioms (a). 4) Every consistent S-system has a model. 5) If a system X has a model, it will have a finite or denumerable model. 6) Let σ_n denote the formula

$$x_1=x_2\vee x_1=x_3\vee\cdots\vee x_1=x_{n+1}\vee x_2=x_3\vee\cdots\vee x_n=x_{n+1},$$

so that σ_n is valid in a model $\mathfrak{M}$ if and only if $\mathfrak{M}$ has no more than n elements. If σ_n does not belong to the O-system X for any n, then for every cardinal $\mathfrak{m}$ there is a model $\mathfrak{M}$ of the system X of power at least equal to $\mathfrak{m}$. 7) The principle of ordering (that every set can be ordered) follows from Gödel's completeness theorem for O-systems; that is, from the theorem that every consistent O-system has a model. 8) If no condition σ_n belongs to the S-system X, then for every cardinal number $\mathfrak{m}$ there exists a model $\mathfrak{M}$ of the system X of power equal to or higher than $\mathfrak{m}$. 9) $\mathfrak{M}_0=\langle A_0, S_1, S_2, \cdots, G_0, G_1, \cdots\rangle$ is called a submodel of $\mathfrak{M}=\langle A, R_1, R_2, \cdots, F_0, F_1, \cdots\rangle$ if $0\neq A_0\subseteq A$, S_i is R_i restricted to A_0, and $a_1, \cdots, a_{k_i}\in A_0$ implies $F_i(a_1, \cdots, a_{k_i})\in A_0$. An S-system X is called an S-O system if $\mathrm{Cn}_S(X\cdot O)=X$. We then have: The system X is an S-O system if and only if every submodel of a model of X is also a model of X. 10) A system X is called categorical in the power $\mathfrak{m}$ ($\mathfrak{m}$ being an arbitrary cardinal number), if the system X has a model of power $\mathfrak{m}$, and if any two models of this system of power $\mathfrak{m}$ are isomorphic. It then follows that if the system X has no finite models, and if for some $\mathfrak{m}$, X is categorical in the power $\mathfrak{m}$, then X is a complete system.

L. N. Gál

Ribeiro, Hugo. The notion of universal completeness. Portugal. Math. 15 (1956), 83–86.

A consistent set Σ of sentences of a formal theory T is called universally complete if and only if for any universal sentence ϕ of T either (i) ϕ follows from Σ, or (ii) there is an integer n and a sentence α_n saying that there are at most n elements and such that $\phi\leftrightarrow\alpha_n$ follows from Σ. This paper contains the motivation for this definition and a few applications. The notion will be further discussed in a later paper.

L. N. Gál (Ithaca, N.Y.).

See also: de Groot, p. 790; Povarov, p. 860; Trahtenbrot, p. 860; Kemeny, p. 860.

ALGEBRA

Combinatorial Analysis

See: Hall, p. 816; Khan, p. 816; Kemeny, p. 860.

Linear Algebra

Reiner, Irma. On the two-adic density of representations by quadratic forms. Pacific J. Math. **6** (1956), 753–762.

Using Siegel's notation, the author denotes by $A_q(S, T)$ the number of solutions of $X'SX \equiv T \pmod q$, where S and T are integral matrices of order m and n respectively, in integral matrices X. Now $A_q(S, T)$ has been determined by Siegel and others for q a power of an odd prime and is easily shown to depend on A_2 and A_8 for q a power of 2. The author gives expressions for $A_2(S, T)$ and $A_8(S, T)$ when the determinants of S and T are both odd. These are explicit for the former, and means of getting explicit expressions are given for the latter. *B. W. Jones.*

Albert, A. A.; and Muckenhoupt, Benjamin. On matrices of trace zero. Michigan Math. J. **4** (1957), 1–3.

The authors extend to any field the result of K. Shoda [Jap. J. Math. **13** (1936), 361–365] that if M is a square matrix with vanishing trace then matrices A and B can be found such that $M=AB-BA$. In the proof of the Lemma the statement $b_{i,i+3}=0$ should read $b_{i,j}=0$, $j \geqq i+3$. *D. E. Rutherford* (St. Andrews).

Gauthier, Luc. Commutation des matrices et congruences d'ordre un. Bull. Soc. Math. France **84** (1956), 283–294.

This paper consists of elementary remarks on the geometrical interpretation of the commutativity of matrices. *A. A. Albert* (Chicago, Ill.).

Bing, Kurt. On Sylvester's law of nullity. Amer. Math. Monthly **64** (1957), 100.

The author's words run as follows: "When one extends Sylvester's Law of Nullity from square matrices over a field of rectangular ones, the question arises which are the most general conditions. The answer is as follows. Sylvester's Law of Nullity. The nullity of AB is less than or equal to the sum of the nullities of A and B. It is greater than or equal to the nullity of A; it is greater than or equal to that of B provided A does not have more columns than it has rows." The proof is elementary and in the usual pattern. *A. A. Bennett* (Providence, R.I.).

Khan, N. A. The characteristic roots of the product of matrices. Proc. Indian Acad. Sci. Sect. A. **45** (1957), 84–88.

Kato, Tosio. On the Hilbert matrix. Proc. Amer. Math. Soc. **8** (1957), 73–81.

It is known that the largest eigenvalue λ_n of the Hilbert matrix $((i+k-1)^{-1})$ or $((i+k)^{-1})$ $(i, k=1, \cdots, n)$ tends to π for $n \to \infty$ [see O. Taussky, Quart. J. Math. Oxford Ser. **20** (1949), 80–83; MR **11**, 16]. It is also known that the infinite Hilbert matrix does not admit π as an eigenvalue when the corresponding eigenvector is assumed to be in the Hilbert space l^2. The question was raised whether π is an eigenvalue when the restriction on the eigenvector is dropped [see O. Taussky, Bull. Amer. Math. Soc. **60** (1954), 290]. This question is answered here affirmatively for a wide class of matrices which contains the Hilbert matrix as a special case. The eigenvector which corresponds to π is obtained as the limiting vector of the sequence of eigenvectors corresponding to λ_n. The ith components of the latter (when normalized) form a monotone converging sequence for each i. *O. Taussky-Todd* (Washington, D.C.).

Bulatović, Zarija. On a new method of reduction of the general equation of conics to the canonical form. Bull. Soc. Math. Phys. Serbie **8** (1956), 61–64. (Serbo-Croatian. English summary)

★ **Hieke, Max. Vektoralgebra.** Mathematisch-Naturwissenschaftliche Bibliothek, 4. B. G. Teubner Verlagsgesellschaft, Leipzig, 1956. vi+154 pp. DM 9.20.

This volume is basically a textbook in the algebra of vectors in Euclidean 3-space. Material is also included concerning vectors in 4-space, and cartesian tensors in 3-space and 4-space. The standard topics such as addition, linear independence, and scalar and vector products are included; but, in addition, there is a discussion of the invariance of these concepts under the orthogonal group. There are also many applications to physics.
C. B. Allendoerfer (Seattle, Wash.).

Janekoski, V. Sur les démonstrations de quelques théorèmes en algèbre des vecteurs. Bull. Soc. Math. Phys. Serbie **8** (1956), 65–72. (Serbo-Croatian. French summary)

See also: Beaumont, p. 789; Čul'ik, p. 792; Vučkovic', p. 801; Parameswaran, p. 801; Kowalski, p. 804; Nef, p. 810; Hill, p. 812; Nagler, p. 825; Szabó, p. 838; Kemeny, p. 860.

Polynomials

Rahman, Q. I. On the zeros of a class of polynomials. Proc. Nat. Inst. Sci. India. Part. A. **22** (1956), 137–139.

This paper is concerned with two theorems on the zeros of a polynomial $P(z)=a_0+a_1z+\cdots+a_nz^n$. (1) If $|a_0|+|a_1|+\cdots+|a_{n-1}| \leqq n|a_n|$, $2^m r^n(n+1)|a_n/a_0|<1$, $m=\frac{1}{2}(n+1)$ and $2r \geqq 1$, then at least M zeros of P lie outside the circle $|z|=r$. (2) Let $N(x)$ denote the number of zeros of P in $|z| \leqq x$, $x>0$, and let $R=\max|a_{n-j}/a_n|^{1/j}$ for $j=1, 2, \cdots, n$; if $\min|a_j| \geqq 1$ for $j=0, 1, \cdots, n$, and $\max|a_j| \geqq |a_n|$ for $j=0, 1, \cdots, n-1$, then

$$N(R/K) \leqq q\{\log|(n+1)a_nR^n|\}/\log K$$

for every real $K>1$. Theorems (1) and (2) were proved for $M=1$ and $q=2$ by S. K. Singh [same Proc. **19** (1953), 601–603; MR **15**, 524]. In the present paper they are proved with $M=\frac{1}{2}(n+1)$ or $M=1+\frac{1}{2}n$ according as n is odd or even, and with $q=1$. These improvements on Singh's results are established by use of Jensen's Formula.
M. Marden (Milwaukee, Wis.).

Kuipers, L. Note on the location of zeros of polynomials. III. Simon Stevin **31** (1957), 61–72.

This paper contains a number of theorems about the location of the zeros of polynomials of the form

$$F(z)=Af(z+a)+Bf(z+b),$$

where A, B, a, and b are given (complex) constants and $f(z)$ is an nth degree polynomial with the given zeros z_j, $j=1, 2, \cdots, n$. The following are examples of the theorems. (1) Given the constants t_k ($|t_k|>1$, $k=1, 2, \cdots, n$) and $t=t_1t_2\cdots t_n$. Let R denote the intersection of the interiors of the n circles $|z-z_k+h|=|t||z-z_k-h|$ ($k=1, 2, \cdots, n$). Then $F(z)=f(z+h)+tf(z-h)\neq 0$ in R. (2) Let R be the intersection of the interiors of the hyperbolas comprised of all points that satisfy simultaneously the inequalities $|z-z_j+h|>|z-z_j-h|+|t|$ ($j=1, 2, \cdots, n$). Then if a is complex with $|a|=1$,

$$F(z)=f(z+h-t)+af(z-h)\neq 0$$

in R. (3) Let $h=|h|e^{iu}$, $k=-|k|e^{iu}$, $|k|\geq|h|$. Let L be a line making an angle of $u+(\pi/2)$ with the positive real axis and let R be the half plane which is bounded by L and which contains the points $z+h$ corresponding to points z on L. Then, if the zeros of $f(z)$ lie in R, the same is true of the zeros of $F(z)$ and $G(z)$, where

$$F(z)=f(z+h)+af(z-h),$$

$|a|=1$, and $G(z)=\int_{z+k}^{z+h} f(v)dv$. Applications are also made to certain trigonometric integrals. The author's results are generalizations of some results due to Obrechkoff [Tôhoku Math. J. **38** (1933), 93–100], Weisner [ibid. **44** (1937), 175–177], Sz. Nagy [ibid. **41** (1936), 415–422] and de Bruijn [Duke Math. J. **17** (1950), 197–226; MR **12**, 250]. The proofs of the theorems are based largely upon elementary use of inequalities.

M. Marden.

Parodi, Maurice. Sur quelques propriétés de polynomes. Bull. Sci. Math. (2) **80** (1956), 76–81.

Given a polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$ ($a_n\neq 0$) for which $R^2=|a_2|+|a_3|+\cdots+|a_n|>1$ and $|a_1|>2R$. The author shows that $f(z)$ has one and only one zero in the disk $|a_1+z|\leq R$. To prove this theorem, he introduces the matrices A and $H=C^{-1}AC$, where A has $f(z)=0$ as its characteristic equation and $C=\|c_i\delta_{ij}\|$, with $c_i=1$ for $i=1, 2, \cdots, n-1$ and $c_n=R$, and applies a theorem due to A. Brauer [Duke Math. J. **13** (1946), 387–395; MR **8**, 192]. Also, by applying a theorem due to J. L. Walsh [Proc. Nat. Acad. Sci. U.S.A. **8** (1922), 139] the author shows that the kth derivative of $f(z)$ has $n-k-1$ zeros in the unit circle, and one in the circle with center at $-n^{-1}(n-k)a_1$ and with radius of 1 provided $|a_n|>\max\{2n, 1+R^2\}$.

M. Marden (Milwaukee, Wis.).

van de Vooren-van Veen, J. On the number of irreducible equations of degree n in GF(p) and the decomposability of the cyclotomic polynomials in GF(p). Simon Stevin 31 (1957), 80–82. (Dutch)

After proving that the number of irreducible equations of degree n with coefficients in GF(p) is $n^{-1}\sum_{d|n}\mu[n/d]p^d$, by use of "superelements" of GF(p^n), namely those elements of GF(p^n) but not of any GF(p^m) where $m|n$, the author establishes the following criterion: For h a given multiple of p, let n be the least exponent for which $p^n\equiv 1 \pmod{h}$. Then the cyclotomic polynomial Φ_h is reducible into $\phi(h)/n$ factors of degree n.

A. A. Bennett.

Markovitch, D. Sur un mode de factorisation approximative des polynômes. Bull. Soc. Math. Phys. Serbie 8 (1956), 53–58. (Serbo-Croatian. French summary)

Theory of Invariants

See: Hanin, p. 837.

Partial Order Structures

Vilhelm, Václav. The selfdual kernel of Birkhoff's conditions in lattices with finite chains. Czechoslavak Math. J. **5(80)** (1955), 439–450. (Russian. English summary)

A lattice L_0 is called cyclic if there exists the least element O and the greatest element I of L_0 and two chains of L_0 between O and I whose sum is L_0 and whose intersection (in the sense of the theory of sets) contains only O, I. A sublattice $\bar{L}$ of a lattice L is called saturated if any chain between two elements of $\bar{L}$ which is maximal in $\bar{L}$ is maximal in L too. Let further L be a lattice with finite chains. A pair of maximal chains of L with common endpoints is said to have the Jordan-Hölder property if the chains have the same length and if there exists a one-to-one correspondence between the prime quotients of both chains so that the corresponding prime quotients are projective. If every pair of maximal chains which form a cyclic sublattice of length ≥ 3 has the Jordan-Hölder property, then this is true for every pair of maximal chains with common endpoints. Birkhoff's covering condition (ξ'') [G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, Chap. V, § 2; MR **10**, 673] is not valid in L if and only if one of the following conditions is fulfilled: (A) There exists a saturated sublattice of L whose graph is given. (B) There exists a saturated cyclic sublattice of L whose length is ≥ 3. By the selfdual kernel of Birkhoff's conditions the negation of the condition (B) is meant.

M. Novotný.

Jakubík, Ján. Direct decompositions of completely distributive complete lattices. Czechoslovak Math. J. **5(80)** (1955), 488–491. (Russian. English summary)

Let $\{L_i\}$ ($i\in M$) be a set of lattices. The set of all functions $h(i)$ defined on M and having the property that $i_0\in M$ implies $h(i_0)\in L_{i_0}$ becomes a lattice by putting $(h_1\vee h_2)(i)=h_1(i)\vee h_2(i)$ and $(h_1\wedge h_2)(i)=h_1(i)\wedge h_2(i)$ for every $i\in M$. This lattice is called the direct union of the factors L_i. A complete lattice is isomorphic with the direct union of all its directly indecomposable factors if and only if the center of L is a complete completely distributive lattice. This is true if L is a complete completely distributive lattice.

M. Novotný (Brno).

Musti, Romolo; e Buttafuoco, Ettore. Sui subreticoli distributivi dei reticoli modulari. Boll. Un. Mat. Ital. (3) 11 (1956), 584–587.

It is shown that a subset $X=\{x_1, \cdots, x_n\}$ of a modular lattice generates a distributive sublattice if and only if every proper subset of X generates a distributive lattice and

$$\left(\sum_{i=1}^{r} x_i\right)\cdot\prod_{j=r+1}^{n} x_j=\sum_{i=1}^{r}\left(x_i\cdot\prod_{j=r+1}^{n} x_j\right)$$

for $r=2, \cdots, n-1$. This improves on a result due to the reviewer [Proc. Amer. Math. Soc. **6** (1955), 682–688; MR **17**, 341].

B. Jónsson (Minneapolis, Minn.).

Kalicki, J. On the axioms of Grau's ternary algebra. Proc. Leeds Philos. Lit. Soc. Sci. Sect. **6** (1952), 12–13.

A. A. Grau [Bull. Amer. Math. Soc. **53** (1947), 567–572;

MR 9, 3] characterized Boolean algebras in terms of complementation and the ternary operation

$$a^bc=ab+bc+ca.$$

His axioms were $a^b(c^de)=(a^bc)^d(a^be)$, $a^bb=b$, $b^ba=b$, $a^bb'=a$, $b'^ba=a$. It is shown here that the third axiom follows from the others, and that the remaining axioms are independent. *B. Jónsson.*

See also: Jaffard, p. 790.

Rings, Fields, Algebras

Tiago de Oliveira, J. Demonstration élémentaire d'existence de modules et anneaux de caractéristique et cardinalité quelconques. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1956), 361–363.

The author notes that if c is an infinite cardinal, then a direct sum of c copies of the ring of integers modulo k has cardinal c and characteristic k. *M. F. Smiley.*

Peremans, W. Free algebras with an empty set of generators. Nederl. Akad. Wetensch. Proc. Ser. A. **59** = Indag. Math. **18** (1956), 565–570.

Let Q be a consistent set of conditional equations. The notions of a free product and of a tensor product of Q-algebras are defined, and it is shown that these products always exist and are unique up to isomorphism. The author finally considers the problem of embedding a given Q-algebra isomorphically in a Q'-algebra (where $Q\subseteq Q'$ and Q' involves additional constants), and shows that the existence of one such embedding implies the existence of a "universal" embedding. *B. Jónsson.*

Foster, Alfred L. On the finiteness of free (universal) algebras. Proc. Amer. Math. Soc. **7** (1956), 1011–1013.

[For terminology see Foster, Math. Z. **62** (1955), 171–188; MR **17**, 452.] It is proved that if $\mathfrak{P}$ is a functionally strictly complete algebra of order n, and S is the set of all identities which hold in $\mathfrak{P}$, then the n^k-th power of $\mathfrak{P}$ is a free algebra with k generators for the class of all models of S. *B. Jónsson* (Minneapolis, Minn.).

Foster, Alfred L. The generalized Chinese remainder theorem for universal algebras; subdirect factorization. Math. Z. **66** (1956), 452–469.

[For terminology see, e.g., Foster, Math. Z. **62** (1955), 171–188; MR **17**, 452.] It is conjectured that every class $\tilde{\mathfrak{P}}$ of pairwise non-isomorphic primal algebras is a primal cluster. The main theorem verifies this conjecture under the assumption that every finite subclass of $\tilde{\mathfrak{P}}$ is coframal. *B. Jónsson* (Minneapolis, Minn.).

Cohen, I. S.; and Zariski, Oscar. A fundamental inequality in the theory of extensions of valuations. Illinois J. Math. **1** (1957), 1–8.

Es sei K ein bewerteter Körper, mit Bewertung v. Es sei ferner K^* eine endlich-algebraische Erweiterung von K, und $v_1^*, \cdots, v_g^*$ die sämtlichen verschiedenen Fortsetzungen von v auf K^*. Es wird die folgende fundamentale Ungleichung beweisen:

$$(1) \qquad \sum_{i=1}^{g} e_if_i\leq n.$$

Hierbei bedeutet n den Grad von K^* übner K; ferer ist f_i der Restklassengrad von v_i^* über v (d.h. der Grad des Restklassenkörpers Γ_i^* von v_i^* über dem Restklassenkörper Γ von v); ferner ist e_i die Verzweigungsordnung von v_i^* über v (d.h. der Index der Wertgruppe Δ_i^* von v_i^* über der Wertgruppe Δ von v). Die Ungleichung (1) ist wohlbekannt, falls Δ den Rang 1 im Sinne der Theorie der geordneten Gruppen besitzt (dann ist Δ isomorph zu einer Untergruppe der additive Gruppe der reellen Zahlen). Das Hauptgewicht bei den Untersuchungen der Verfasser liegt also auf denjenigen Bewertungen, deren Wertgruppen einen höheren (eventuell unendlichen) Rang besitzen. Die wesentliche Schwierigkeit, die dabei zu überwinden ist, liegt darin begründet, daß die v_i^* bei höherem Rang nicht notwendig unabhängig sind. Es werden daher neben v gleichzeitig auch diejenigen Bewertungen v' von K betrachtet, deren Bewertungsringe $R_{v'}$ den Bewertungsring R_v echt enthalten (dann schreibt man: $v'<v$). Diese v' bilden eine linear geordnete Menge $L(v)$. Bezeichnet man mit $g(v')$ die Anzahl der verschiedenen Fortsetzungen von v' auf K^*, so gilt der folgende, zum Beweis von (1) entscheidende Hilfssatz: Wenn $L(v)$ kein größtes Element besitzt, so gilt

$$g(v)=\max(g(v');\, v'\in L(v)).$$

Zum Schluß wird noch eine hinreichende Bedingung dafür angegeben, daß in (1) das Gleichheitszeichen gilt: Bedeutet R^* die ganzabgeschlossene Hülle von R_v in K^*, so besagt diese Bedingung, daß R^* ein endlicher R_v-Modul ist. Falls Δ vom Range 1 und diskret ist, so ist diese Bedingung auch notwendig. *P. Roquette* (Hamburg).

Hoehnke, Hans-Jürgen. Nilpotenzkriterien. Math. Ann. **132** (1957), 404–411.

Various "radicals" have been defined for a ring, each generalizing certain features of the maximal nilpotent ideal of the classical Wedderburn structure theory. The author is interested in restrictions which will insure the nilpotency of certain of these ideals. He develops criteria, involving weak chain conditions, which not only are sufficient to insure that an ideal A of a ring R is nilpotent (nil) but also are necessary for A to be nilpotent (nil). *W. E. Deskins* (East Lansing, Mich.).

Scott, W. R. On the multiplicative group of a division ring. Proc. Amer. Math. Soc. **8** (1957), 303–305.

Hua [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 533–537; Acad. Sinica Sci. Record **3** (1950), 1–6; MR **11**, 155; **12**, 584] has shown: Let K be a division ring, Z its center, K^*, Z^* the multiplicative groups of K and Z respectively; then (1) K^*/Z^* has a trivial center, and (2) K^* is not solvable. The present author proves, amongst other things: If K is a non-commutative division ring, then K^*/Z^* has no non-trivial normal abelian subgroups. This result clearly contains those of Hua. He also extends a result of the reviewer [Proc. Amer. Math. Soc. **7** (1956), 1021–1022; MR **18**, 557] by showing that a noncentral element in K has $o(K)$ conjugates, where $o(K)$ is the number of elements in K. *I. N. Herstein.*

Sas, F. [Szász, F.]. On rings such that every subring is a direct summand of the ring. Mat. Sb. N.S. **40**(82) (1956), 269–272. (Russian)

A ring R is said to have property P, if any subring of R is a direct summand (in the ring-theoretical sense) of R. A ring R has property P if and only if it is a (discrete) direct sum of rings of prime orders, and moreover any such decomposition of R contains for a given prime p at

most one direct summand which is a prime field of characteristic p.

{The reviewer should like to point out that the proof of Lemma 1. is not correct.} *A. Kertész* (Debrecen).

Leavitt, W. G. Modules without invariant basis number. Proc. Amer. Math. Soc. **8** (1957), 322–328.

Continuing a previous paper [same Proc. **7** (1956), 188–193; MR **17**, 1048], the author shows that for every integer $n>1$ there exists a ring without zero divisors over which a finitely based module has invariant basis number if and only if it has a basis of length less than n.

R. E. Johnson (Northampton, Mass.).

Beaumont, Ross A. Matric criteria for the uniqueness of basis number and the equivalence of algebras over a ring. Publ. Math. Debrecen **4** (1956), 469–480.

Let R be an associative ring with identity, and let ${}_mR_n$ be the set of all $m\times n$ matrices with elements in R. A matrix $A\in {}_mR_n$ is a unit if there exists $B\in {}_nR_m$ such that $AB=I_m$ and $BA=I_n$. Let $N(R)$ be a free (left) module with basis $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$. The following version of a theorem of Everett [Bull. Amer. Math. Soc. **48** (1942), 312–316; MR **3**, 262] is obtained. The elements $\eta_1, \cdots, \eta_m$ are a basis for $N(R)$ if and only if there exists a unit $A\in {}_mR_n$ such that $\eta'=A\varepsilon'$, where $\eta'=(\eta_1, \cdots, \eta_m)$ and $\varepsilon'=(\varepsilon_1, \cdots, \varepsilon_n)$. It follows that every free R-module with finite basis has a unique basis number if and only if ${}_mR_n$ contains no units for all m, n with $m\neq n$, and in particular if R is commutative, or if R satisfies the maximal condition for right or left ideals.

If multiplication (not necessarily associative) be defined for elements $\alpha=\sum r_i\varepsilon_i$ and $\beta=\sum s_j\varepsilon_j$ of $N(R)$ by $\alpha\cdot\beta=\sum r_is_j\gamma_{jk}{}^i\varepsilon_k$, where $\gamma_{jk}{}^i\in R$ and $i, j, k=1, 2, \cdots, n$, then $N(R)$ is said to be a (left) algebra over R, which is denoted by $N=[N(R), \varepsilon_i, \Gamma]$, where Γ is the $n^2\times n$ multiplication matrix of N defined by

$$\Gamma=\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \vdots\\ \Gamma_n\end{pmatrix}, \quad \Gamma_i=(\gamma_{jk}{}^i).$$

If $M=[M(R), \eta_k, \Delta]$ is another (left) algebra over R, where $k=1, 2, \cdots, m$ and Δ is the multiplication matrix for the η's, then the author proves that M and N are isomorphic if and only if there exists a unit $A\in {}_mR_n$ such that $\Delta=(A\otimes A)\Gamma B$, where $A\otimes A$ is the Kronecker product and $AB=I_m$ as above. If in particular $\varepsilon_1, \cdots, \varepsilon_n$ are closed under multiplication and form a groupoid G, the above result is applied to groupoid algebras to obtain the following criterion for the isomophism of G with a groupoid H with elements $\eta_1, \cdots, \eta_n$, viz, $G\cong H$ if and only if there exists a permutation matrix P of degree n such that $(P\otimes P)\Gamma P^{-1}=\Delta$, where Γ, Δ are the multiplication matrices of G and H. *S. A. Jennings.*

Dixmier, J. Sous-algèbres de Cartan et décompositions de Levi dans les algèbres de Lie. Trans. Roy. Soc. Canada. Sect. III. (3) **50** (1956), 17–21.

Let L be a finite-dimensional Lie algebra over a field F of characteristic 0, let R be the radical of L, and let H be a Cartan subalgebra of L. It has been shown by Chevalley [Théorie des groupes de Lie, t. III, Hermann, Paris, 1955; MR **16**, 901] that there exists a maximal semisimple subalgebra S of L such that H is the sum of $H\cap R$ and a Cartan subalgebra K of S. Here, the author proves the following additional facts: (1) The decomposition $H=K+H\cap R$, with K a Cartan subalgebra of some (not prescribed) maximal semisimple subalgebra of L, is unique. (2) $[K, H\cap R]=(0)$, and $H\cap R$ is a Cartan subalgebra of the centralizer of K in R. (3) If S is a maximal semisimple subalgebra of L, K a Cartan subalgebra of S, and C a Cartan subalgebra of the centralizer of K in R, then $K+C$ is a Cartan subalgebra of L. (4) Suppose that F is algebraically closed. Then, to every given Cartan subalgebra H of L, one can find a maximal semisimple subalgebra S of L such that every root space of L with respect to H is the sum of its intersection with S and its intersection with R. Also, if S is any given maximal semisimple subalgebra of L, and K is a Cartan subalgebra of S, then C of (3) can be so chosen that every root space of L with respect to the Cartan subalgebra $K+C$ is the sum of its intersection with S and its intersection with R.

G. P. Hochschild (Princeton. N.J.).

Cartier, Pierre. Théorie différentielle des groupes algébriques. C. R. Acad. Sci. Paris **244** (1957), 540–542.

The author gives an intrinsic definition of the notion of hyperalgebra over an arbitrary field K: it is an algebra A over K, with unit, which is a direct sum of $K\cdot 1$ and of a two-sided ideal A^+, and in which is defined a homomorphism Δ of A into $A\otimes A$ satisfying the following: 1) $(\Delta\otimes 1)\circ\Delta=(1\otimes\Delta)\circ\Delta$; 2) $\Delta(A)$ consists of symmetric elements; 3) $\Delta(a)-a\otimes 1-1\otimes a\in A^+\otimes A^+$ for $a\in A^+$. He shows that this notion includes as special case the notion of hyperalgebra of a formal Lie group, introduced by the reviewer [Comment. Math. Helv. **28** (1954), 87–118; MR **16**, 12 and Amer. J. Math. **77** (1955), 218–244; MR **16**, 789], and shows how the latter's theorems can be extended to his more general hyperalgebras, especially the homomorphism theorem and the relation between commutative hyperalgebra and certain types of modules over the ring of Witt-vectors over K. He announces finally his intention to apply this machinery to the solution of some outstanding problems in the theory of abelian varieties over a field of characteristic $p>0$, and states a theorem which puts strong restictions on the structure of the hyperalgebra of a commutative formal Lie group, when that group is the "local" group corresponding to an algebraic group. No proofs are given. *J. Dieudonné.*

See also: Croisot, p. 790; Steinfeld, p. 790; Lewis, p. 793; Peterson, p. 815.

Groups, Generalized Groups

Szász, F. A characterization of the cyclic groups. Rev. Math. Pures Appl. **1** (1956), no. 2, 13–16.

For a non-negative integer k, define G^k to be the group generated by the kth powers of elements of a group G. The author proves that cyclic groups G are characterized by the property that, for every cyclic subgroup H, $H=G^k$ for some k. [For earlier papers on this result see Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 491–492; Acta. Math. Acad. Sci. Hungar. **6** (1955), 475–477; Publ. Math. Debrecen **4** (1956), 237–238; MR **17**, 709, 940; **18**, 187]. *S. A. Jennings* (Vancouver, B.C.).

Rosati, Luigi Antonio. Sui gruppi ogni sottogruppo ciclico dei quali è caratteristico. Boll. Un. Mat. Ital. (3) **11** (1956), 544–552.

A group G has property P if every cyclic subgroup of

G is characteristic. Such a group G is shown to be abelian, and if G is the direct product of a finite number of cyclic factors (and in particular if G is of finite order) then G is cyclic. A discussion of finite torsion groups with property P is given, and a theorem of F. Szász [see the preceding review] is obtained. *S. A. Jennings.*

Wagner, Daniel H. On free products of groups. Trans. Amer. Math. Soc. 84 (1957), 352–378.

The principal result states that if f is a homomorphism of a free group F onto a free product $\prod_{i\epsilon I}^{*} H_i$, then F possesses a free decomposition $F=\prod_{i\epsilon I}^{*} J_i$ such that $f(J_i)=H_i$ for every $i \epsilon I$. This was proved by Gruschko [Mat. Sb. N.S. **8(50)** (1940), 169–182; MR **2**, 215] under the assumption that F be finitely generated. Another special case, where the groups H_i are free, was obtained by Federer and the reviewer [Trans. Amer. Math. Soc. **68** (1950), 1–27; MR **11**, 323]; the methods employed here follow rather closely the ideas of that paper, although considerable additional complications arise in the more general situation. *B. Jónsson.*

Gaschütz, Wolfgang. Zu einem von B. H. und H. Neumann gestellten Problem. Math. Nachr. **14** (1955), 249–252 (1956).

The main theorem states: If F is a free group with n generators, G is a group with n generators, and N is a finite normal subgroup of G, then every homomorphism of F onto G/N is induced by a homomorphism of F onto G. Corollaries: If M and N are finite normal subgroups of a homogeneous group G, then every isomorphism between G/M and G/N is induced by an automorphism of G. If $G=G_1\times\cdots\times G_n$ is finite, then G is generated by n elements if and only if each of the groups $G_1, \cdots, G_s$, $G_1/R(G_1)\times\cdots\times G_s/R(G_s)$ is generated by n elements, where $R(G_j)$ is the intersection of the maximal proper normal subgroups of G_j. *B. Jónsson.*

de Groot, J. Indecomposable abelian groups. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 137–145.

The author constructs a family of 2^c additive groups of real numbers such that no element of the family can be mapped homomorphically (and non-degenerately) into any other element of the family.

He also shows that there exist 2^c (mutually non-isomorphic) additive groups of real numbers of rank $\mathfrak{c}$ such that the automorphism group of each of them is trivial.

The author proves several other theorems and calls attention to the following two unsolved questions: a) Does there exist an indecomposable abelian group of power $>\mathfrak{c}$? b) Does there exist for every infinite cardinal $\mathfrak{m}$ a family of $2^{\mathfrak{m}}$ mutually non-isomorphic torsion-free abelian groups? (He remarks that for non-abelian groups the answer is affirmative.)

{Remark: The author attributes to the reviewer a paper whose author is J. Erdös.}

P. Erdös (Haifa).

★ **Taketa, Kiyosi. Über die Struktur der metabelschen Gruppen.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 257–259. Science Council of Japan, Tokyo, 1956.

This paper appeared in fuller detail in J. Math. Soc. Japan 7 (1955), 491–529 as a continuation of a series of three previous articles. The pertinent references are given in the review of the 1955 paper [MR **18**, 377].

Jaffard, Paul. Réalisation des groupes complètement réticulés. Bull. Soc. Math. France **84** (1956), 295–305.

Lorenzen showed [Math. Z. **45** (1939), 533–553; MR **1**, 101] that every lattice-ordered abelian group G is order-isomorphic with a subdirect product G' of totally ordered abelian groups, G' being a sublattice of the product. Pteviously [J. Math. PuresAppl. (9) **32** (1953), 203–280; MR **15**, 284], the author of this note considered the problem of the irreducibility of this realization; i.e., whether or not one of the factors in the product is superfluous. In this paper, the question is raised: If G is a complete lattice-ordered abelian group, can G be realized as a complete sublattice of a product of totally ordered abelian groups? The following answer is given. A necessary and sufficient condition that such a group G admit this realization is that G admit an irreducible realization, in the above sense. Further, if G admits such a realization, it is essentially unique, and the totally ordered groups are order-isomorphic either to the additive group of integers or to the additive group of reals. The paper concludes with an application to the lattice-ordered group of real-valued functions on a compact space. *F. B. Wright.*

Croisot, R. Applications résiduées. Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 453–474.

Let E be an ordered set and let P be the demi-group of all mappings of E into E. Let ξ, η belong to P. A partial ordering in P is defined by $\xi\leqq\eta$ if $x\xi\leqq x\eta$ for all x in E. It follows that $\xi\leqq\eta$ implies $\zeta\xi\leqq\zeta\eta$ for all ζ in P. Restricting attention to the set P' of isotonic mappings σ ($x\leqq y$ implies $x\sigma\leqq y\sigma$) it then follows that in P', $\xi\leqq\eta$ implies also that $\xi\zeta\leqq\eta\zeta$. The identity mapping is denoted by ε and belongs to P'. An isotonic mapping α is said to be residuated if for all x in E there exists an element z in E such that $z\alpha\leqq x$ and the set of all such elements z contains a greatest one which is denoted by $x\beta$. Then β is an isotonic mapping of E into E and, for all x in E, $x\leqq x\alpha\beta$ and $x\beta\alpha\leqq x$. The mapping β is called the residual mapping of α. If the ordering of E is reversed, then β is an isotonic residuated mapping of E whose residual mapping is α. The isotonic residuated mappings α of E form a demi-group P_1' and the set of residual mappings β, a demi-group P_2'. The correspondence $\alpha\leftrightarrow\beta$ is an anti-isomorphism of P_1' and P_2'. If E is an ordered set of right ideals X in a demi-group D, then $X\rightarrow XD$ is an isotonic residuated mapping and $X\rightarrow X\cdot.D$ is its residual mapping. Various properties of isotonic residuated mappings in general are obtained, including conditions that a mapping generated by α and β be the identity mapping. Special cases in which $\alpha^2\leqq\alpha$ or $\alpha\leqq\varepsilon$ are then dealt with. Finally applications to the case of mappings of right ideals in a demi-group are considered. *D. C. Murdoch* (Vancouver, B.C.).

Steinfeld, Otto. Über die Quasiideale von Halbgruppen. Publ. Math. Debrecen **4** (1956), 262–275.

Let H be a (multiplicative) semigroup. A subset $\mathfrak{a}$ of H is said to be a quasi-ideal of H if $H\mathfrak{a}\frown\mathfrak{a}H\subseteqq\mathfrak{a}$. [This concept for rings was introduced earlier by the author, Acta Math. Acad. Sci. Hungar. **4** (1953), 289–298; **6** (1955) 479–484; MR **16**, 992; **17**, 1180]. It is shown that the intersection of a right ideal R and a left ideal L of H is a quasi-ideal, and conversely, every quasi-ideal $\mathfrak{a}$ can be written as such an intersection, viz. $\mathfrak{a}=(\mathfrak{a}, H\mathfrak{a})\frown(\mathfrak{a}, \mathfrak{a}H)$. If R and L are both minimal, then their intersection is a minimal quasi-ideal of H, and conversely, every minimal quasi-ideal can be written in this way. Relationships between minimal quasi-ideals and minimal ideals (right,

left and two-sided) of H are studied. Typical of the results obtained are the following. A quasi-ideal $\mathfrak{a}$ of H is a group if and only if it is minimal, in which case it can be written in the form $\mathfrak{a}=\varepsilon H\varepsilon$ where ε is the unit element of $\mathfrak{a}$; and all such minimal quasi-ideals are isomorphic. If H has at least one minimal quasi-ideal, the union of all minimal quasi-ideals of H is the intersection of all two-sided ideals of H. Finally, minimal quasi-ideals in semi-groups with relative inverses, and in completely simple semi-groups in the sense of Rees [Proc. Cambridge Philos. Soc. 36 (1940), 387–400; MR 2, 127] are discussed.

S. A. Jennings (Vancouver, B.C.).

See also: Scott, p. 788; Nakayama, p. 793.

THEORY OF NUMBERS

General Theory of Numbers

Schinzel, A. Generalisation of a theorem of B.S.K.R. Somayajulu on the Euler's function $\varphi(n)$. Ganita 5 (1954), 123–128 (1955).

The author proves that the set of numbers $\varphi(n+1)/\varphi(n)$ is dense in $(0, \infty)$.

P. Erdős (Haifa).

Mitrinovitch, Dragoslav S. Sur une question d'analyse diophantienne. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. 6 (1956), 4 pp. (Serbo-Croatian summary)

In a previous paper [Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire 1 (1948), 49–95; MR 10, 527] the author proposed the problem of finding all sets of rational numbers a_ν, p_ν ($\nu=1, 2, 3, \cdots, m$) such that the sum $\sum_{k=1}^{n} \prod_{\nu=1}^{m} (a_\nu+(k-1)p_\nu)$ could be written as a product of $m+1$ factors, linear in n and with rational coefficients. In the present paper some new sets of solutions are given in case $m=2$, each set containing an infinity of numbers a_1, a_2, p_1, p_2.

W. Ljunggren (Blindern).

Hunter, John. A note on integer solutions of the Diophantine equation $x^2-dy^2=1$. Proc. Glasgow Math. Assoc. 3 (1956), 55–56.

The purpose of this note is to establish a procedure for obtaining solutions of the Fermat equation (1) $x^2-dy^2=1$ by Newton's method of approximating to the root $\sqrt{d}$ of the equation $f(x)\equiv x^2-d=0$. If x_1 is any positive rational number, then Newton's method gives a sequence (x_n) of rational numbers defined by the relation

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=\frac{x_n^2+d}{2x_n}\ (n\geqq 1).$$

It is shown that each x_1 provides a sequence of positive rational solutions (p_n, q_n), $n=2, 3, \cdots$, of (1), which are such that

$$(2)\qquad p_{n+1}+q_{n+1}\sqrt{d}=(p_2+q_2\sqrt{d})^m\ (m=2^{n-1}),$$

where

$$p_2=\frac{x_1^2+d}{|x_1^2-d|},\quad q_2=\frac{2x_1}{|x_1^2-d|}.$$

If p_2 and q_2 are integers, then (2) gives a sequence of positive integer solutions of (1). Putting $d=m^2+r$, where $r\neq 0$ and $-m+1\leqq r\leqq m$, $m>0$, the author then gives a new proof of the well-known theorem that p_2, q_2 give the minimum positive integer solution of (1) in the following two cases: $r=-1$, $x_1=m-1$; and r divides $2m$, $r\neq -1$, $x_1=m$.

W. Ljunggren (Blindern).

Stolt, Bengt. On a Diophantine equation of the second degree. Ark. Mat. 3 (1957), 381–390.

For solving an equation of the type $x^2-Dy^2=\pm N$, where D and N are positive rational integers, one may use either the theory of quadratic forms or the theory of quadratic fields. T. Nagell has shown [e.g. Introduction to number theory, Wiley, New York, 1951, pp. 195–212; MR 13, 207] how it is possible to determine all solutions in rational integers x and y independently of these theories and completely elementarily. His investigations have been continued by the author [e.g. Ark. Mat. 2 (1952), 1–23; MR 14, 247]. In this paper the following equation is considered: (1) $Au^2+Buv+Cv^2=\varepsilon N$, where A, B, C and N are rational integers, $A>0$, $N>0$, $\varepsilon=\pm 1$, and where $B^2-4AC=D$ is a positive integer which is not a perfect square. It is shown how it is possible to avoid the usual linear transformations and congruences in order to obtain all the integral solutions of (1). If $u=t/A$ is a fractional number and v is an integer which satisfy (1), the number $f(u, v)=(2Au+Bv+\sqrt{D})/2$ is called a solution of (1). The set of all solutions associated with each other forms a class of solutions of (1). If u and v are two integers satisfying (1), $f(u, v)$ is called an integral solution of (1). It is proved that if one solution of a class K is an integral solution, then every solution of K is integral. If $f(u, v)$ is the fundamental solution (in a sense defined) of (1), then one has the following inequality:

$$0<v\leqq\frac{AN}{D}(x_1-2\varepsilon)^{\frac{1}{2}},$$

where $v=0$ also may be possible if $\varepsilon=1$, (x_1, y_1) denoting the fundamental solution of $x^2-Dy^2=4$. If in $f(u, v)$ the number $(2Au+Bv)v/N$ is an integer, the class is said to be quasi-ambiguous. Some theorems concerning such classes are proved. Finally, examples are given.

W. Ljunggren (Blindern).

Simmons, H. A. Classes of maximum numbers associated with two symmetric equations in N reciprocals. Proc. Amer. Math. Soc. 8 (1957), 169–175.

In the present paper the author generalizes results obtained in a series of previous papers [e.g. Bull. Amer. Math. Soc. 48 (1942), 295–303; MR 3, 269]. Let $\sum_{i,j}(1/x)$ stand for the elementary symmetric function of the jth order of the i reciprocals $1/x_p$, $p=1, 2, \cdots, i$, $i>0$, with $\sum_{i,j}(1/x)\equiv 0$ when $i<j$ or $j<0$, and $\sum_{i,j}(1/x)\equiv 1$ when $j=0$. Let further

$$(1)\qquad \varphi_p\left(\frac{1}{x}\right)=\sum_{p,r}\left(\frac{1}{x}\right)+\sum_{k=r+1}^{s} L_k\sum_{p,k}\left(\frac{1}{x}\right)\ (r\leqq p\leqq n).$$

At first the author considers the equation

$$(2)\qquad \varphi_n\left(\frac{1}{x}\right)=\frac{b}{a},\ a=(c+1)b-1,$$

in which r, s, n are positive integers with $r<s\leqq n$; each L_k is a non-negative integer; and b, c are arbitrary positive integers. The solution $x=(x_1, x_2, \cdots, x_n)$ of (2), obtained by minimizing the variables $x_1, x_2, \cdots, x_{n-1}$ in this order, one at time, among the positive integers is called the Kellogg solution ω, while an E-solution is

one in which

$$(3) \qquad x_1 \leqq x_2 \leqq x_3 \leqq \cdots \leqq x_n,$$

where $x_1, x_2, \cdots, x_{n-1}$ are positive integers. Further a solution of (2) is called admissible if the elements x_j in x are real numbers $\geqq 1$, satisfying (3), and such that

$$(4) \qquad \varphi_p\left(\frac{1}{x}\right) \leqq \frac{b}{a} - \frac{1}{ax_1x_2\cdots x_p} \quad (p=r, \cdots, n-1).$$

When x is an E-solution of (2), (4) holds for each indicated value of p. Furthermore, one readily finds that when $x=\omega$, the equality sign applies in (4) for each value of p. Being an E-solution of (2), ω is admissible. Relative to (2) the following theorem is proved. If x is an admissible solution and $x\neq\omega$, then $x_n<\omega_n$ and $P(x)<P(\omega)$, where $P(x)\equiv P(x_1, x_2, \cdots, x_n)$ denote any nonconstant, symmetric polynomial in $x_1, x_2, \cdots, x_n$ with no negative coefficients. Finally, a similar result is proved for an equation obtained by multiplying the first term in the right member of (1) by L_r, L_r being a positive integer, in case $a=b=1$. *W. Ljunggren* (Blindern).

Bini, Umberto. Il teorema di Waring e la rappresentazione per cubi di un numero. Archimede 8 (1956), 172–176.

Schinzel, André. Sur un problème concernant la fonction $\varphi(n)$. Czechoslovak Math. J. 6(81) (1956), 164–165. (Russian summary)

The author gives a very simple proof of the following theorem of Pillai: For every k these exists integers n_k so that $\varphi(x)=n_k$ has more than k solutions. *P. Erdös.*

Čulík, Karel. Remarque sur un problème de K. Zarankiewicz. Acta Acad. Sci. Čechoslovenicae Basis Brunensis 27 (1955), 341–348. (Czech. Russian and French summaries)

Let M be the set of m-row, n-column matrices $A_n{}^m(k)$ with exactly k zero entries. Such a matrix is said to have property $z(i, j)$ if it has a submatrix $P_j{}^i(ij)$, that is, one with i rows and j columns and all zero entries. Let K be the set of all k for which each matrix in M has $z(i, j)$. The problem of Zarankiewicz is to determine the numerical function $Z_{ij}(m, n)=\min k$ for k in K. Z is the smallest number k of zero entries for which all $A_n{}^m(k)$ have $z(i, j)$. The author's main result is that the values of Z may be determined solely in terms of matrices in M which have only 0 or 1 as entries, and are such that if r_x (s_y) is the number of zeros in row x (column y), then $r_1\geqq r_2\geqq\cdots\geqq r_m$ and also $s_1\geqq s_2\geqq\cdots\geqq s_n$. *V. E. Beneš.*

See also: Lewis, p. 793.

Analytic Theory of Numbers

Sierpinski, W. Sur les ensembles de nombres naturels qui ont un nombre fini d'éléments communs avec toute leur translation. Ganita 5 (1954), 137–141 (1955).

A strictly increasing sequence $E=\{p_1, p_2, p_3, \cdots\}$ of positive integers is said to possess property P if $p_{n+1}-p_n\to\infty$ as $n\to\infty$. This is equivalent to the property that E, regarded as a set of points on a straight line, has at most a finite number of points in common with each translation of E. If $\psi_E(x)$ is the number of points of such a set E which do not exceed x, it is shown that $x^{-1}\psi_E(x)\to 0$ as $x\to\infty$ when E has property P. Also, if E is any infinite set of positive integers, sets of positive integers H, Q not possessing property P exist such that $\psi_H(x)/\psi_E(x)\to 0$ and $\psi_Q(x)/\psi_E(x)\to 1$ as $x\to\infty$; further, a set K possessing property P exists such that $\psi_K(x)/\psi_E(x)\to\infty$ as $x\to\infty$. Thus from the fact that $\psi_E(x)$ increases rapidly one cannot conclude that E does not possess property P. An application to the case where p_n is the nth prime is given. {Confusion may arise on p. 139, where $E(\alpha)$ is used to denote the integral part $[\alpha]$ without explanation. On line 9 of p. 141 read $\leqq k$ for $\leqq 1$.} *R. A. Rankin* (Glasgow).

Turan, Paul. On the zeros of the zeta-function of Riemann. J. Indian Math. Soc. (N.S.) 20 (1956), 17–36.

Denote by $N(\alpha, T)$ the number of roots of $\zeta(s)=0$ in the parallelogram $\alpha\leqq\sigma\leqq 1$, $0<t\leqq T$. Carlson and Hoheisel proved that

$$N(\alpha, T)<c_1T^{4\alpha(1-\alpha)}(\log T)^{c_2},$$

and Ingham proved this with 8/3 instead of 4. The author discusses the so called density hypothesis

$$(1) \qquad N(\alpha,T)<c_1T^{2\alpha(1-\alpha)}(\log T)^{c_2}$$

or its slightly weaker form

$$(2) \qquad N(\alpha, T)<c_1T^{(2+\epsilon)\alpha(1-\alpha)}.$$

(It is known that (1) and (2) would imply $p_{n+1}-p_n<p_n{}^{\frac{1}{2}+\epsilon}$.) Ingham proved that Lindelöf's hypothesis $|\zeta(\frac{1}{2}+it)|=o(t^\epsilon)$ implies (2), and the author found a different proof by his well-known method; further he proved (2) "near" to the line $\sigma=1$. More precisely, he proved that for a certain (small) c_3 and $T>c_4$, $N(\alpha, T)<T^{2\alpha(1-\alpha)+(1-\alpha)^{1.4}}$ $(1<c_{21}<\alpha<1)$. The author finally discusses several conjectures which seem weaker than Lindelöf's hypothesis and which imply (2). Here I want to state only one of them: There exists a $g(x)$, $\lim_{x=0} g(x)=0$, $g(x)$ monotone increasing and positive for $x>0$, which has the following property. Assume that for some α_4 $(\frac{1}{2}<\alpha_4<1)$, $\zeta(s)$ does not vanish in the parallelogram $\alpha_4\leqq\sigma\leqq 1$, $|t-\tau|\leqq\log\tau$ with a $\tau\geqq 3$. Then we have for $0<\delta<1/10$, $\tau>c_{28}(\delta)$, in the parallelogram $\alpha_4-2\delta\leq\sigma\leq\alpha_4$, $|t-\tau|\leq\delta$, the inequality $|\zeta(s)|\leqq\tau^{g(\delta)(\alpha_4-\sigma)}$. The author proves (2) using this conjecture and discusses possible methods which might lead to the proof of this conjecture. *P. Erdös* (Haifa).

Postnikov, A. G. Generalization of one of the Hilbert problems. J. Indian Math. Soc. (N.S.) 20 (1956), 207–216.

Let m be a fixed positive integer. The author proves that the $\varphi(m)$ Dirichlet L-series $L(s, \chi_1), \cdots, L(s, \chi_{\varphi(m)})$ are difference-differentially independent. That is, if $k_1, \cdots, k_q$ are distinct real numbers, and if N is an integer $\geqq 0$, there does not exist any algebraic identity between the $(N+1)q\varphi(m)$ functions $L^{(j)}(s+k_i, \chi_h)$ $(j=0, \cdots, N;$ $i=1, \cdots, q;$ $h=1, \cdots, \varphi(m))$. As an application it is shown that there is no algebraic relation between the various partial derivatives (with respect to x and s) of the $\varphi(m)$ functions

$$L(x, s, \chi_h)=\sum_{n=1}^{\infty}\chi(n)n^{-s}x^n \quad (h=1, \cdots, \varphi(m)).$$

The case $m=1$ was one of Hilbert's problems (Paris 1901); it was solved by D. D. Morduchai-Boltovskoy [Izvestiya Warsawskovo Polytechnicheskovo Instituta 1914; Tôhoku Math. J. 35 (1932), 19–34] and by A. Ostrowski [Math. Z. 8 (1920), 241–298]. The present author follows Ostrowski's method. *N. G. de Bruijn* (Amsterdam).

Postnikov, A. G. On Dirichlet *L*-series with the character modulus equal to the power of a prime number. J. Indian Math. Soc. (N.S.) **20** (1956), 217–226.

The author considers Dirichlet L-series $L(s, \chi)$, where χ is a primitive character mod p^n, and p is a prime >2. Both p and n are large, and submitted to $n^Q \geq \log p \geq C_1 n^4 (\log n)^3$ (C_1, C_2, C_3, Q are arbitrary positive constants, $Q>4$). It is shown that $L(s, \chi)$ has no zeros in the region $|t|>C_2$, $\sigma>1-C_3(\log p^n)^{-Q/(Q+1)}(\log \log p^n)^{-1}$, provided that n is large enough.

The proof uses summation of $\chi(1+pu)$ with respect to u. It is shown that $\chi(1+pu)$ can be written as $\exp(2\pi i f(u))$, where $f(u)$ is a polynomial, so that Vinogradov's estimates on trigonometric sums can be applied.

There are several minor errors and many misprints.

N. G. de Bruijn (Amsterdam).

Sastry, S.; and Singh, Raghuraj. A problem in additive number theory. J. Sci. Res. Banaras Hindu Univ. **6** (1955–56), 251–265.

The authors prove that almost all positive integers can be expressed as $x_1^2+x_2^3+x_3^3+x_4^6$, and that all large integers can be written as $x_1^2+x_2^3+x_3^3+x_4^6+\sum_{s=5}^{45} x_s^{s+1}$. The proofs closely follow those of K. F. Roth, who proved a similar result [Proc. London Math. Soc. (2) **53** (1951), 381–395; MR **13**, 14]. *N. G. de Bruijn.*

Wright, E. M. Partitions of multi-partite numbers. Proc. Amer. Math. Soc. **7** (1956), 880–890.

Let $F_j(Y)=\prod(1+X_1^{k_1}X_2^{k_2}\cdots X_j^{k_j}Y)$, the product being extended over all non-negative $k_1, \cdots, k_j$. The author studies the coefficients $R_j(n)$ and $Q_j(n)$ defined by $F_j(Y)=\sum_{n\geq 0} R_j(n) Y^n$ and $\{F_j(-Y)\}^{-1}=\sum_{n\geq 0} Q_j(n) Y^n$. The case $j=1$ is well-known, and Bellman [Bull. Amer. Math. Soc. **61** (1955), 92, problem 3] has asked for a formula for $Q_2(n)$. Let $\beta_j(m)=(1-X_1^m)\cdots(1-X_j^m)$. Then

$$Q_j(n)=\sum_{(n)} \prod (h_m!)^{-1}\{m\beta_j(m)\}^{-h_m},$$

the sum extending over all partitions of n of the form $n=\sum mh_m$, and the product over all the different parts m in the partition. There is a similar formula for $R_j(n)$. $P_j(n)=\beta_j(1)\cdots\beta_j(n)Q_j(n)$ is a polynomial of degree $n(n-1)/2$ in each X_i (for $j>1$) with integral coefficients which the author conjectures are non-negative. Results for special values of the X_i and recursion formulas are obtained. The cases $n=2$ and $n=3$ are worked out explicitly, and the case $j=2$ is discussed briefly. Finally, the author obtains an asymptotic formula for $Q_2(n)$, for fixed X_1, X_2 such that $|X_i|<1$. *N. J. Fine.*

Theory of Algebraic Numbers

Lewis, D. J. Cubic congruences. Michigan Math. J. **4** (1957), 85–95.

It has been conjectured that every cubic form in at least 10 variables over an algebraic number field represents 0 in a non-trivial way. As a step in this direction the author proves the Theorem 3: If Γ is a finite extension of the field of rational numbers, if Δ is the ring of algebraic integers in Γ, and if $\mathfrak{m}$ is an ideal in Δ, then every congruence $\alpha_1x_1^3+\cdots+\alpha_nx_n^3\equiv 0 \pmod{\mathfrak{m}}$, where $n>7$, $\alpha_i \in \Delta$, has a solution in Δ which is non-trivial modulo each prime factor of $\mathfrak{m}$. This result is derived from Theorem 2: If K is a complete field under a non-archimedean valuation and has a finite residue class field k, then every equation $\alpha_1x_1^3+\cdots+\alpha_nx_n^3=0$, where $\alpha_i \in K$, has a non-trivial solution if $n\geq 7$, but there need not be a solution if $n=6$. Here the difficult case is when k has the characteristic 3, and then the construction of the solutions becomes very involved. *K. Mahler.*

Nakayama, Tadasi. On modules of trivial cohomology over a finite group. Illinois J. Math. **1** (1957), 36–43.

The author's main results on the topic of the title have been announced previously [Proc. Japan Acad. **32** (1956), 373–376; MR **18**, 191]. Several proofs are presented here, involving a number of special criteria for a G-module A, where G is a p-group, to be of trivial cohomology, i.e., to be such that $H^r(K, A)=(0)$, for all integers r and all subgroups K of G. The various special criteria and the main result finally yield the following general criterion: let G be a finite group, and let A be a G-module. Let

$$(0)\rightarrow A_1\rightarrow A_0 \leftarrow A\rightarrow(0)$$

be an exact G-module sequence, with A_0 free over the integral group ring $Z(G)$. For every prime p dividing the order g of G, let G_p be a p-Sylow subgroup of G. Then A is of trivial cohomology if and only if, for every p, A_1/pA_1 is free over the group algebra $(Z/pZ)(G_p)$. Moreover, if A is g-torsion free, then A may take the place of A_1 in this criterion. *G. P. Hochschild* (Princeton, N.J.).

ANALYSIS

Functions of Real Variables

Bögel, K. Die Struktur der stetigen Funktionen einer Veränderlichen. I, II. J. Reine Angew. Math. **196** (1956), 1–33, 137–154.

A result obtained for differentiable curves by J. Hjelmslev [Danske Vid. Selsk. Forhandlinger, Oversigt, 1911, 433–494] was essentially generalized by O. Haupt [J. Reine Angew. Math. **164** (1931), 50–60] as follows: Every plane parametric curve $(x=x(t),\ y=y(t))$ is the union of at most denumerably many segments, convex arcs, primitive arcs of infinite order, and their points of accumulation. The author gives a detailed investigation of the fine structure of the primitive arcs of infinite order, but only for plane curves of the form $y=f(x)$ (defined and finite in a closed interval J). Two preparatory chapters discuss some set theoretical foundations and the so-called "triangular discriminant" of $y=f(x)$, which decides whether the point triple (*) (x_i, y_i), (x_j, y_j), (x_k, y_k) is convex, concave, or linear. The principal results are contained in the third and fourth chapter. An interior point x_j of J is called a "linear-, or convex-, or concave-point" of f if in a certain neighborhood U of x_j for every pair x_i, x_k of U with $x_i<x_j<x_k$ the point triple (*) is linear, or convex, or concave, respectively. (E.g. the point $x=0$ is a "convex-point" of $f(x)=x^2(1+\sin^2 x^{-1})$ for $x\neq 0$, $f(0)=0$.) The other interior points of J are called "remainder points" ("Restpunkte") by the author, and the set of these remainder points is designated by $R(f, J)$. The convex- and concave-points of f and the endpoints of the considered interval are called "principal points" ("Hauptpunkte") of f. In the following, f is always assumed to be continuous in J. The principal points form an F_σ-set; hence the linear-points and remainder points together

form a G_δ-set. The main result of the investigation is the statement that $\bar{J}$ can be subdivided into disjoint subintervals of two types (whose complement is nowhere dense in $\bar{J}$), namely intervals to which $R(f, J)$ is nowhere dense and intervals to which $R(f, J)$ is dense. In particular, for a nowhere differentiable function f, $R(f, J)$ is dense in $\bar{J}$. If the set $R(f, J)$ is dense in $\bar{J}$, then $R(f, J)$ is a residual set, the set $\Lambda(f, J)$ of the linear-points is empty, the set $H^+(f, J)$ of the convex-points and the set $H^-(f, J)$ of the concave-points are also dense in $\bar{J}$, but of first category. Consider f in an open interval $J_1 \subset J$; the author calls f "purely linear" in J_1 if $\Lambda(f, J_1)=J_1$, "purely convex" in J_1 if $H^+(f, J_1)=J_1$, "purely concave" in J_1 if $H^-(f, J_1)=J_1$. In each of these cases he calls f "regular" in J_1. Moreover, he calls f "piecewise regular" ("abschnittweise regulär") in J_1 if every interval $J_2 \subset J_1$ contains a subinterval J_3 in which f is regular. Then f is piecewise regular in J_1 if and only if $R(f, J)$ is nowhere dense to J_1.

The fifth chapter forms part II of this memoir (the sixth chapter will be published as part III in J. Reine Angew. Math. **197**). It is shown that, using details of the structure discussed in part I, one can generate all continuous functions by a unique method of construction based on a particular type of uniform approximation; hereby every continuous function is obtained only once. It follows from this construction that, for instance, the set of the principal points can be prescribed as an arbitrary F_σ-set. *A. Rosenthal* (Lafayette, Ind.).

Tarnawski, E. On the spaces of functions satisfying Dini's condition. Fund. Math. **43** (1956), 141–147.

Let the functions denoted by $f(x)$ be defined for every real x, continuous, and periodic with period 1. Let $w(t)$, $w_1(t)$ be non-decreasing and different from zero for $t>0$, and tending to zero for $t\to 0$. Moreover, set

$$W(\tau)=\int_\tau^1 \frac{dt}{w(t)} \quad \text{and} \quad W_1(\tau)=\int_\tau^1 \frac{dt}{w_1(t)}.$$

Let D_w denote the space of all functions $f(x)$ satisfying Dini's generalized condition, i.e. the inequality

$$\int_0^1 \frac{|f(x+t)-f(x)|}{w(t)}\,dt \leq 1$$

for every x. Let S be a set of functions $f(x) \in D_w$ and satisfying, for every x, the condition

$$(*) \qquad \int_0^1 \frac{|f(x+t)-f(x)|}{w_1(t)}\,dt=\infty.$$

Then the author proves that, under certain conditions, S is a residual set in D_w. These conditions are:

$$(1) \qquad \lim_{\tau\to 0+} W_1(\tau)=\infty,$$

$$(2) \qquad \int_0^1 \frac{t}{w(t)}\,dt<\infty, \quad \int_0^1 \frac{t}{w_1(t)}\,dt<\infty,$$

$$(3) \qquad \lim_{t\to 0+} W(t)\cdot\frac{w(t)}{t}>1,$$

$$(4) \qquad \limsup_{t\to 0+} \frac{w_1(2t)}{w_1(t)}<\infty,$$

$$(5) \qquad \lim_{t\to 0+} \frac{w_1(t)}{w(t)}=0.$$

Moreover, one can replace D_w by the space C of all continuous and periodic functions $f(x)$, namely: If $w_1(t)$ satisfies the assumptions (1) and (4), then the set S of functions satisfying the condition (*) for every x is residual in the space C. *A. Rosenthal* (Lafayette, Ind.).

Ostrowski, Alexander. Mathematische Miszellen. XXV. Über das Verhalten von Iterationsfolgen im Divergenzfall. Jber. Deutsch. Math. Verein. **59** (1956), Abt. 1, 69–79.

Let $f(x)$ be a continuous function defined in a (closed or open or half-open) interval J whose value also belong to J, and consider the sequence of iterations

$$(*) \qquad x_{\nu+1}=f(x_\nu) \quad (\nu=0, 1, 2, \cdots; x_0 \in J).$$

S. L. Leibenson [Prikl. Mat. Meh. **17** (1953), 351–360; MR **14**, 1072] discussed the case of divergence of these sequences. The author now gives simpler proofs of these results [see (1)] under more general conditions and other theorems [see (2)] of a similar type. In particular, he proves the following theorems: (1) For $x_\nu \in J$ let $l=\liminf x_\nu$, $L=\limsup x_\nu$ belong to J and be different. Then (α) there is at least one fixed point of (*) in (l, L); (β) $F_1(x)\equiv f(x)-x$ again and again takes on positive values in each right-hand neighborhood of l and negative values in each left-hand neighborhood of L; (γ) if all fixed points of (*) are repelling, then there is at least one conjugate pair of points of (*) in $[l, L]$. Here two numbers u, $v \in J$ are called conjugate points of (*) if $f(u)=v$, $f(v)=u$, $u\neq v$. (2) Let z be a right-hand attractive fixed point and Z a left-hand attractive fixed point of (*) with $z<Z$. Then there is at least one other fixed point of (*) between z and Z. *A. Rosenthal* (Lafayette, Ind.).

Marcus, Solomon. Sur une généralisation des fonctions de G. Hamel. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 584–589.

Una funzione reale $f(x)$ della variabile reale x è interna nell'intervallo (a, b) se per $a<x<y<b$, quando non risulta $\min(f(x), f(y))<f((x+y)/2)<\max(f(x), f(y))$, risulta

$$\min(f(x), f(y))=f((x+y)/2)=\max(f(x), f(y)).$$

E l'Autore studia le funzioni interne che non siano al tempo stesso monotone. Funzioni interne non monotone son, per esempio, le funzioni di Hamel, cioè le soluzioni discontinue dell'equazione funzionale $f(x+y)=f(x)+f(y)$. L'Autore prova che la totalità delle funzioni di Hamel non esaurisce quella delle funzioni interne non monotone; che il diagramma di una funzione interna e non monotona può anche non esser ovunque denso nel piano, mentre quello di una funzione di Hamel è notoriamente ovunque denso nel piano; che una funzione interna e non monotona è sprovvista della proprietà di Baire rispetto ad ogni insieme di seconda categoria, provvisto della proprietà di Baire e contenuto nell'intervallo (a, b) in cui la funzione è assegnata; che una funzione interna e non monotona, definita nell'intervallo (a, b), non è qualitativemente continua in nessun punto di (a, b). L'Autore studia le funzioni interni e non monotone anche dal punto di vista della connessione e della totale mancanza di connessione dei loro diagrammi. *G. Scorza Dragoni* (Padova).

Newns, W. F. A note on rectifiable curves. Edinburgh Math. Notes no. **40** (1956), 12–14.

Let f be a continuous complex-valued function on a real interval (a, b), whose real and imaginary parts are of bounded variation. If s is arc length along the range of f, it is shown that if f is absolutely continuous in (α, β) then $s(\beta)-s(\alpha)\leq\int_\alpha^\beta |f'(t)|dt$. This simple lemma is used to

obtain two known results: I. (Tonelli) For any rectifiable curve $\int_\alpha^\beta |f'(t)|dt \leq s(\beta)-s(\alpha)$, equality holding for all α, β ($a \leq \alpha < \beta \leq b$) if and only if f is absolutely continuous in (a, b); II. (Pollard) If f' exists in some neighborhood of t_0 and if $|f'|$ is upper semi-continuous at t_0, then s is differentiable at t_0 and $s'(t_0)=|f'(t_0)|$. Various extensions are indicated. *E. A. Coddington.*

Nikol'skiĭ, S. M. Boundary properties of functions in regions with angles. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 26–28. (Russian)

A summary of further results along the lines of the author's paper [Mat. Sb. N.S. **40(82)** (1956), 303–318; MR **18**, 723]. *M. G. Arsove* (Seattle, Wash.).

Nikol'skiĭ, S. M. On the Dirichlet problem for regions with corners. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 33–35. (Russian)

Results similar to those of the paper reviewed above and of the paper cited there, but for functions solving the Dirichlet problem on G for prescribed boundary functions ϕ. *M. G. Arsove* (Seattle, Wash.)

Froda, Alexandre. Propriétés des fonctions réelles sur des réseaux continus de "para-voisinages". C. R. Acad. Sci. Paris **243** (1956), 549–552.

Let E be a set of the n-dimensional Euclidean space E^n ($n>1$) and let $\rho>0$ be a given constant. In a preceding note [same C. R. **242** (1946), 1948–1951; MR **17**, 953] the author defined the family $\mathfrak{F}\equiv\mathfrak{F}(\rho, E)$ of star-like "para-voisinages" $\mathfrak{J}\equiv\mathfrak{J}(p)$ of $p \in E$. If E is dense in E^n and $\mathfrak{J}(p)$ is defined for every $p \in E$, then $\mathfrak{F}$ is called a "net" ["réseau"] of E. A suitable metric is introduced in $\mathfrak{F}$ and so the notion of continuity of $\mathfrak{F}$ is defined. The set E is called "relatively dense" in E^n if there exists a constant $r>0$ such that E has (at least) one point in every closed sphere $\overline{S^n}$ with radius r. If this is the case for each $r>s$, then E is said to be "relatively dense (s)" in E^n. If the single-valued function $f(p)$ is bounded on every $\mathfrak{J} \in \mathfrak{F}$, then f is called "bounded (ρ) on $\mathfrak{F}$". Let $M\equiv M(f, p, \mathfrak{J})$ and $m\equiv m(f, p, \mathfrak{J})$ be the upper and lower bound, respectively, of f on $J(p)$. Then the author proves the following theorem: If $f(p)$ is bounded (ρ) on the continuous net $\mathfrak{F}$, then the sets $\{p|f(p)\leq M\}$, $\{p|f(p)\geq m\}$, $\{p|f(p)<M\}$, $\{p|f(p)>m\}$ are relatively dense (4ρ) in E^n. Another result is obtained even if $\mathfrak{F}$ is not assumed to be continuous. *A. Rosenthal* (Lafayette, Ind.).

Lipiński, J. S. Sur les ensembles $\{f(x)>a\}$, où $f(x)$ sont des fonctions intégrables au sens de Riemann. Fund. Math. **43** (1956), 202–229.

The author discusses the sets $\{f(x)>a\}$ for certain classes of functions f, in whose characterization always Riemann integrability is used. He proves the following theorems: (1) In order that a point set E of the space E^n can be represented as $E=\{f(x)>a\}$ where f is a Riemann integrable function in every interval of E^n, it is necessary and sufficient that there exists an F_σ-set E^* with $E\subset E^*\subset \bar{E}$ and $|E^*\cdot \mathrm{Fr}\, E|=0$. (2) In order that a linear point set E can be represented as $E=\{f(x)>a\}$ where f is a Riemann integrable derivative in every interval, it is necessary and sufficient that $E \in M_4$ [see Z. Zahorski, Trans. Amer. Math. Soc. **69** (1950), 1–54, in particular p. 3; MR **12**, 247] and $|E\cdot \mathrm{Fr}\, E|=0$. Z. Zahorski [loc. cit.] already characterized the sets $\{f(x)>a\}$ where f is a bounded derivative; his result is used here by the author. (3) In order that a linear point set E can be represented as $E=\{f(x)>a\}$, where f is approximately continuous and Riemann integrable in every interval, it is necessary and sufficient that E is an F_σ-set every point of which is a point of density and $|E\cdot \mathrm{Fr}\, E|=0$. In connection with (1) the author proves also the following theorems which characterize Riemann integrability of f by means of the sets $\{f(x)>a\}$, namely: In order that the set of discontinuities of a function f defined in E^n be of measure zero, it is necessary and sufficient that one of the following 3 conditions be satisfied: either (a) $|\{f(x)>a\}\cdot \mathrm{Fr}\{f(x)>a\}|=0$ and $|\{f(x)<a\}\cdot \mathrm{Fr}\{f(x)<a\}|=0$ for every number a; or (b) $|\mathrm{Fr}\{f(x)>a\}\cdot \mathrm{Fr}\{f(x)>b\}|=0$ for every pair of numbers a, b ($a\neq b$); or (c) the set of numbers a for which $|\mathrm{Fr}\{f(x)>a\}|>0$ is at most denumerable. (c) is also used in the proof of (3). *A. Rosenthal* (Lafayette, Ind.).

Austin, D. G. A Lipschitzian characterization of approximately differentiable functions. Portugal. Math. **15** (1956), 19–29.

Let f denote a finite, measurable function defined on the N-dimensional Euclidean space E_N. The main result of this paper is the following theorem: If the absolute values of the approximate partial derivates of f are less than P on a bounded set K, then for every $\eta>0$ there exist disjoint sets $S_i\subset K$ ($i=1, 2, \cdots, p$) such that f is Lipschitzian on S_i with the Lipschitz constant P and $m(K-\bigcap_{i=1}^p S_i)<\eta$. The converse of this result is contained in the following theorem: Let the measurable function f on E_N be given. If for some measurable set K and for the arbitrary $\eta>0$ there exists a set $G\subset K$ with $m(G)<\eta$ and a function g which is Lipschitzian on E_N and coincides with f on $K-G$, then f has finite approximate partial derivatives and an approximate differential almost everywhere on K. Some further applications of the main theorem are also given. *A. Rosenthal.*

Romanovskiĭ, P. I.; and Ignat'ev, U. V. On a generalization of the idea of differential of higher order. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **21** (1954), 35–48. (Russian)

The authors first discuss inclusion relations between the classes of functions having right-hand nth order derivatives at a point x_0, of functions having limits of nth order right-hand difference quotients at x_0, and of functions admitting nth degree polynomial approximation at x_0.

Generalizing the last of the foregoing classes, relative to any system of functions $\varphi_i(h)$, $i=1, \cdots, n$, such that $\varphi_i(h)>0$ for $h>0$ and such that $\varphi_{i+1}(h)=o[\varphi_i(h)]$ as $h\rightarrow+0$, they consider the class of functions $F(x)$ admitting representations of the form

$$F(x_0+h)-F(x_0)=c_1\varphi_1+\cdots+c_n\varphi_n+o[\varphi_n],$$

where the c_i are suitable constants depending on the particular function $F(x)$.

In a further generalization, n is replaced by a continuous parameter through the device of replacing the functions $\varphi_i(h)$ by a function $\varphi(h, t)$ such that $\varphi(h, t)>0$ for $h>0$, $t>0$ and such that, for constants $\alpha<\beta$, $\varphi(h, \alpha)=o[\varphi(h, \beta)]$ as $h\rightarrow+0$. They then consider the class of functions $F(x)$ admitting representations, through Lebesgue-Stieltjes integrals, of the form

$$F(x_0+h)-F(x_0)=\int_0^\alpha \varphi(h, t)d\psi(t)+o[\varphi(h, \alpha)],$$

where $\psi(t)$ is a suitable function depending on the particular function $F(x)$.

The existence and certain properties, applications, and further extensions of these classes of functions are discussed. *E. F. Beckenbach* (Los Angeles, Calif.).

Inoue, Masao. Dirichlet problem relative to a family of functions. J. Inst. Polytech. Osaka City Univ. Ser. A. 7 (1956), 1–16.

The author continues the postulational approach of L. K. Jackson and the reviewer [Pacific J. Math. 3 (1953), 291–313; MR 14, 1084] to the theory of subfunctions for a dominating family of functions of more than one variable, and to the study of a Dirichlet problem relative to that family. Applications are made to the study of solutions of partial differential equations of elliptic type, of the form

$$\partial^2u/\partial x^2+\partial^2u/\partial y^2=f(x, y, u).$$

E. F. Beckenbach (Los Angeles, Calif.).

See also: Inoue, p. 796; Fedorov, p. 796.

Measure, Integration

Mařík, Jan. Bemerkungen zur Theorie des Oberflächenintegrals. Czechoslovak Math. J. 6(81) (1956), 387–400. (Russian. German summary)

Let E_m be euclidean m space. In this paper, a surface is a mapping φ, continuous with continuous partial derivatives, of an open $G\subset E_{m-1}$ into E_m. φ is called associated with a set $A\subset E_m$ if for every $b\in G$ there is a neighborhood $U\subset G$ of b and $\varepsilon>0$ such that for $t\in U$, $0<\alpha<\varepsilon$, we have $\varphi(t)-\alpha w^\varphi(t)\in A$ and $\varphi(t)+\alpha w^\varphi(t)\in A$, where $w^\varphi(t)$ is the outer product of the $m-1$ partial derivative vectors of $\varphi(t)$.

Let H be the boundary of A. Let G_n $(n=1, 2, \cdots)$ be a sequence of open sets in E_{m-1} and φ_n $(n=1, 2, \cdots)$ mappings of G_n associated with A such that the sets $\varphi_n(G_n)$ are pair-wise disjoint and the projections of the set $H-\bigcup_{n=1}^\infty \varphi_n(G_n)$ into the coordinate hyperplanes all have $(m-1)$-dimensional measure zero.

Let v be a continuous mapping of the closure $\bar{A}$ of A into E_m satisfying the conditions: a) The partial derivatives exist, except on a set negligible in a sense specified in the paper, and their integrals over A are finite. b) For every $\varepsilon>0$, there is a $C>0$ such that for all $z\in A$ with $|z|>C$ we have $|v(z)|<\varepsilon$.

c) $$\sum_{n=1}^\infty\int_{G_n}|v(\varphi_n(t))|\,|w^{\varphi_n}(t)|dt<\infty.$$

Under these conditions, it is shown that

$$\int_A \operatorname{div} v(z)dz=\sum_{n=1}^\infty\int_{G_n} v(\varphi_n(t))w^{\varphi_n}(t)dt.$$

In the proof, essential use is made of the relation

$$\int_G F(t)|D_\lambda(t)|dt=\int_{E_m}\Phi(x)dx,$$

where $G\subset E_m$ is open, λ is a mapping, continuous with continuous partial derivatives, of G into E_m, $D_\lambda(t)$ is the functional determinant of λ, F is a function on G such that $\int_G F(t)D_\lambda(t)dt$ exists, and $\Phi(x)=\sum_{\lambda(t)=x}F(t)$.

{Remark: The proof appears to be obscure on p. 392 where an isolated set of real numbers is ordered

$$\cdots<y_{-1}<y_0<y_1<\cdots$$

although the size of the interval of isolation seems to depend on the point.} *C. Goffman* (Norman, Okla.).

Besicovitch, A. S. Analysis of tangential properties of curves of infinite length. Proc. Cambridge Philos. Soc. 53 (1957), 69–72.

This brief note deals with three interesting examples of curves of σ-finite length. *L. C. Young.*

See also: Lipiński, p. 795; Besicovitch, p. 813.

Functions of Complex Variables

Fedorov, V. S. Monogenic n-dimensional vector functions. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki 5 (1954), 65–70. (Russian)

Let the vector function $\mathbf{f}=(u^1, \cdots, u^n)$ of the radius vector $\mathbf{r}=(x^1, \cdots, x^n)$ have continuous first-order partial derivatives $u_j^k=\partial u^k/\partial x^j$ in a domain D, and for each point M of D consider the directional-derivative vector $\mathbf{f_a}(M)$ defined by

$$\mathbf{f_a}(M)\equiv\lim\,[\mathbf{f}(M')-\mathbf{f}(M)]/MM'$$

as $M'\to M$ from the direction $\mathbf{a}$. The author calls $\mathbf{f}$ a continuous monogenic function of $\mathbf{r}$ provided that at each point M of D the scalar product $p\equiv\mathbf{a}\cdot\mathbf{f_a}(M)$ is independent of the direction $\mathbf{a}$.

It is shown that for any continuous monogenic function $\mathbf{f}$ the scalar product p has continuous mixed partial derivatives of all orders with respect to the parameters x^j, and that each component function u^j has continuous mixed partial derivatives with respect to the two parameters x^j and x^k, where k is arbitrary. Further, these functions satisfy the following partial differential equations:

$$p_{jj}+p_{kk}=0,\quad u_{jj}^j+u_{kk}^j=0\quad(j\neq k).$$

For $n=2$ it is noted that continuous monogenic functions $\mathbf{f}$ are ordinary monogenic functions of the complex variable $z=x^1+ix^2$, and for $n=3$ it is shown that the scalar product p must be a linear function of the components x^j. *E. F. Beckenbach* (Los Angeles, Calif.).

Ozawa, Mitsuru. On Grötzsch's extremal affine mapping. Kōdai Math. Sem. Rep. 8 (1956), 112–114.

The author proves that the affine mapping of one rectangle onto another minimizes the Dirichlet integral

$$\iint\left\{\left|\frac{\partial S}{\partial z}\right|^2+\left|\frac{\partial S}{\partial \bar{z}}\right|^2\right\}dxdy$$

in the class of mappings S which map corresponding sides onto corresponding sides. *H. L. Royden.*

★ **Pfluger, Albert. Theorie der Riemannschen Flächen.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. xii+248 pp. DM 36.00.

Because of the rapid growth of the theory of Riemann surfaces in the last two decades, a monograph on the subject which takes into account the most recent developments as well as improvements in exposition is to be welcomed. The present volume, whose emphasis is in the direction of the fundamental existence theorems, is a valuable contribution to the monographic literature on Riemann surface theory. It is distinguished by a thorough treatment, modern in language and method, of foundational questions.

The initial three chapters are concerned with fundamental concepts and definitions as well as an account of

methods belonging to more general disciplines such as topology and the theory of differential forms. In the first chapter the notion of a Riemann surface is introduced by the traditional crossing of the Weierstrass-Riemann bridge, that is, by abstracting from the Weierstrassian concept of analytic continuation, as expressed in the notion of an analytisches Gebilde, the notion of a Riemann surface in the sense of Weyl-Radó. In addition to furnishing an important instance of a Riemann surface, this approach sets the stage for the classical realization problems of Riemann and Koebe. An aspect of the realization problem is treated in Chapter Six, where it is shown (for the non-compact case) with the aid of the Weierstrass-Florack theorem concerning analytic functions with assigned zeros that there exists an analytisches Gebilde (with analytic component functions) of given conformal type. Other topics studied in Chapter One are: subharmonic functions including the Perron theorem; the Nevanlinna proof of Radó's theorem concerning the fact that a Riemann surface in the sense of Weyl-Radó has a countable base; the endowing of orientable surfaces (topological, differentiable) with a conformal structure; the triangulation theorem for Riemann surfaces.

Chapter Two is concerned with coverings of Riemann surfaces. In addition to bringing out the relation with analytic continuation, the chapter gives an account of the essential properties of coverings (ramified or not) including such matters as universal covering, the monodromy theorem, the Riemann-Hurwitz theorem. The third chapter entitled "Homology and Cohomology" develops the fundamental notions concerning differential forms and integration. The Dieudonné partition of unity is used (as is now standard) to arrive at the Stokes theorem. Contact with the Hilbert space approach is made by the introduction of the (skew) scalar product of 1-forms. The notion of cohomology for exact 1-forms is introduced via the subspace of total 1-forms. Homology is introduced in terms of integration. The concept of winding number and the allied residue theorem and argument principle find their natural setting here. The equivalence of the integration-theoretic notion of a 1-cycle homologous to zero and the geometric notion of a bounding 1-cycle is established. Homotopy-theoretic simple-connectivity is shown to imply homology-theoretic simple-connectivity. On the other hand, the converse proposition is envisaged as a consequence of either the topological classification of surfaces or the Riemann mapping theorem for homology-theoretic simply-connected Riemann surfaces. With this chapter the preliminaries are disposed of and the way is prepared for a comprehensive study of the existence theorems of the theory, which is a central object of the book.

Existence theorems for harmonic functions and forms are studied in Chapter Four by both the Perron and Dirichlet methods. Not only is the existence aspect considered, but also the problem of constructive methods. The Perron method is employed for the Dirichlet boundary value problem. It is in this framework that the Green's function for a Riemann surface with positive ideal boundary is treated. The Dirichlet principle is treated at length; it is applied to establish the existence of harmonic forms cohomologous to exact 1-forms and the existence of harmonic functions with assigned singularities.

The stage is now set for the central mapping theorems of the theory. These are given in Chapter Five which treats: the Riemann mapping theorem, automorphic groups, the uniformization of an analytisches Gebilde, the mapping theorem for planar surfaces, span and its role in the theory of planar surfaces.

Chapter Six is devoted to harmonic and analytic differentials on both compact and non-compact Riemann surfaces. The first section gives an exposition of the classical theory of abelian differentials on compact Riemann surfaces. Included are such fundamental results as the Riemann period relations, the Riemann-Roch theorem (for integral divisors), the Abel theorem. The next section of the chapter has as its object the study of harmonic differentials of finite norm on non-compact Riemann surfaces. This culminates in an extension of the Riemann period relation for the case of Riemann surfaces belonging to the class O_{HD}. The final section of the chapter gives an account of the Behnke-Stein approximation theorem and its analogue pertaining to harmonic functions which is due to the author. Application of the Behnke-Stein theorem is made to the important problem of the existence of non-constant analytic functions on non-compact Riemann surfaces as well as the existence of analytic differentials belonging to an assigned cohomology class. In addition, the existence of meromorphic functions with assigned principal parts and analytic functions with assigned zeros (Florack) is established by combining consequences of the Behnke-Stein and Pfluger approximation theorems.

The classification problem is the subject of the last chapter. Here a number of the important inclusion relations between the various null classes (O_G, O_{HP}, O_{HB}, O_{AB}, O_{HD}, O_{AD}) are established. A report is given on the extensive literature of the subject. The book terminates with the investigation of criteria which guarantee that a surface be parabolic or hyperbolic. A number of criteria for the parabolic case are given, the conditions being expressed in such terms as extremal length, modules of ring domains, conformal metric, nets of cells with weighted edges. The criterion given for the hyperbolic case is that of Royden. The chapter concludes with a proof of the hyperbolicity of the Schottky covering surfaces of compact Riemann surfaces of genus >1. *M. Heins.*

Oikawa, Kôtaro. A supplement to "Notes on conformal mappings of a Riemann surface onto itself". Kōdai Math. Sem. Rep. **8** (1956), 115–116.

It is shown that the inequality, $N(g, k) \geqq N'(g, k)$, of Theorem 1 of same Rep. **8** (1956), 23–30 [MR **18**, 290] is actually an equality. *M. Heins* (Providence, R.I.).

Wille, R. J. On the integration of Ahlfors' inequality concerning covering surfaces. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 108–111.

It is shown that a slightly weaker form of Nevanlinna's second theorem can be obtained from a similar inequality due to the reviewer [Acta Math. **65** (1935), 157–194]. *L. Ahlfors* (Cambridge, Mass.).

Collingwood, Edward F.; et Lohwater, Arthur J. Inégalités relatives aux défauts d'une fonction méromorphe dans le cercle-unité. C. R. Acad. Sci. Paris **242** (1956), 1255–1257.

Let $f(z)$ be meromorphic in $|z|<1$ with unbounded characteristic function $T(r)$. Let $\Gamma_P(f)$ denote the set of asymptotic values of $f(z)$ approached along paths in $|z|<1$ each of which terminates at a point of $|z|=1$. The authors investigate the class of functions $f(z)$ for which the logarithmic capacity of $\Gamma_P(f)$ is zero. Two theorems

are announced, of which the first is an extension to the hyperbolic case of an earlier result for the parabolic case due to Iversen [Öfversigt af Finska Vetenskaps-societetens Förhandlingar 44 Afdelning A No. 4 (1921)] and the second is a sharpening of an earlier result of Collingwood [Trans. Amer. Math. Soc. **66** (1949), 308–346, p. 336; MR **11**, 94]. *W. Seidel* (Notre Dame, Ind.).

Carleson, Lennart. Representations of continuous functions. Math. Z. **66** (1957), 447–451.

It is shown that if E is a closed set of measure zero on $|z|=1$ and φ is any function continuous on E, then there is an analytic function $f(z)$ in $|z|<1$ which is uniformly continuous in $|z|\leqq 1$ and reduces to φ on E. The key step in the proof is the explicit solution of the problem by means of the Poisson integral formula in the special case where φ assumes only two distinct values. The same result has been derived by Rudin [Proc. Amer. Math. Soc. **7** (1956), 808–811; MR **18**, 472]. A theorem on representation by Fourier-Stieltjes transform of a function continuous on a bounded closed set is also presented. *P. R. Garabedian* (Stanford, Calif.).

Šlionskiĭ, G. G. On the theory of bounded univalent functions. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 962–964. (Russian)

The author announces a number of theorems on bounded univalent functions which generalize known results for arbitrary univalent functions. As a typical example we cite the following: Let $F(\zeta)=\zeta+\alpha_0+\alpha_1/\zeta+\cdots$ be univalent and let $|F(\zeta)|>m$ for $|\zeta|>1$. Let $\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}x_\mu x_\nu$ be a positive quadratic form. If ζ_ν is in $|\zeta|>1$, $\nu=1, 2, \cdots, n$, then

$$\prod_{\nu,\mu=1}^{n}\left|\frac{1-1/\zeta_\nu\bar{\zeta}_\mu}{1-m^2/F(\zeta_\nu)\overline{F(\zeta_\mu)}}\right|^{\alpha_{\nu\mu}} \leqq \prod_{\nu,\mu=1}^{n}\left|\frac{F(\zeta_\nu)-F(\zeta_\mu)}{\zeta_\nu-\zeta_\mu}\right|^{\alpha_{\nu\mu}} \leqq \prod_{\nu,\mu=1}^{n}\left|\frac{1-m^2/F(\zeta_\nu)\overline{F(\zeta_\mu)}}{1-1/\zeta_\nu\bar{\zeta}_\mu}\right|^{\alpha_{\nu\mu}}$$

A. W. Goodman (Lexington, Ky.).

Wintner, Aurel. On the principle of subordination in the theory of analytic differential equations. Acta Math. **96** (1956), 143–156.

Se $f(z, w)$ è una funzione olomorfa delle due variabili complesse z e w, definita nel bicilindro $D(|z|<1, |w|<1)$, col simbolo $\|f\|$ indicheremo il raggio di convergenza dello sviluppo in serie di potenze dell'integrale $w(z)$ del sistema differenziale: $dw/dz=f(z, w)$, $w(0)=0$. Posto $M=\sup|f|$ e supposto $M<+\infty$, è possibile limitare inferiormente $\|f\|$ in funzione di M. Dopo uno studio critico comparative dei risultati precedentemente ottenuti in questo campo a partire dalle classiche ricerche di Cauchy, l'A. dimostra una nuova limitazione. Precisamente, se è: $f=\sum c_{mn}z^m w^n$, $f^*=\sum|c_{mn}|z^m w^n$, si ha $\|f\|\geqq\|f^*\|\geqq 1$ se $M\leqq\frac{1}{4}$, $\|f\|\geqq\|f^*\|\geqq\sin(\pi/4M)$ se $M\geqq\frac{1}{4}$. *C. Miranda* (Napoli).

Sakaguchi, Kōichi. On the multivalency of systems of functions of several complex variables. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 163–173.

Let Z be the vector $\begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}$ and $F(z)$ a vector function $\begin{pmatrix} f_1(Z) \\ \vdots \\ f_l(Z) \end{pmatrix}$. The author investigates sufficient conditions for the p-valency of a suitable derivative of F. For the case $p=1$ his main result yields a known sufficient condition for univalency and for $k=1$ it gives a known sufficient condition for the p-valency of functions of one complex variable. The author also obtains some other sufficient conditions for univalency. *W. T. Martin.*

Eremin, S. A. On bases in the space of analytic functions of two variables. Ukrain. Mat. Ž. **8** (1956), 361–376. (Russian)

The author extends the concepts, methods and results of Markuševič [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **41** (1943), 227–229; Mat. Sb. N.S. **17(59)** (1945), 211–252; MR **6**, 69; **7**, 425] to analytic functions of two variables. *R. P. Boas, Jr.* (Evanston, Ill.).

Hačatryan, I. O. On parametric representation and on certain extremal properties of entire functions of several variables. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **9** (1956), no. 9, 3–14. (Russian. Armenian summary)

The author generalizes the Paley-Wiener theorem on representation of an entire function of exponential type by a finite Fourier transform to functions of several variables. {This has been done before: Plancherel and Pólya, Comment. Math. Helv. **9** (1937), 224–248; **10** (1937), 110–163.} He applies the representation to obtain integral inequalities for the partial derivatives of such functions. *R. P. Boas, Jr.* (Evanston, Ill.).

See also: Rahman, p. 786; Kuipers, p. 786; Parodi, p. 787; Ahiezer, p. 802; Cartan, p. 823; Buchwald, p. 838.

Geometrical Analysis

★ **Brand, Louis. Vector Analysis.** John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1957. xiv+282 pp. $6.00.

Quoting from the Preface, "this book was designed as a short course to give a beginning student the tools of vector algebra and calculus and a brief glimpse beyond into their manifold applications". The nine chapter headings follow: Vector Algebra, Line Vectors, Vector Functions of One Variable, Differential Invariants, Integral Theorems, Dynamics, Fluid Mechanics, Electrodynamics, Vector Spaces. In the final chapter, such topics are considered as linear dependence, bases, the Gram-Schmidt Process, Hilbert Space. An appendix and a detailed index are included.

★ **Barrett, Leonard L. Engineering applications of vector analysis.** The National Press, Palo Alto, California, 1956. xii+114 pp. $3.50.

"The principal claim made for this book is that of simplicity. Persons with elementary knowledge of calculus and electricity should be able to understand it easily .... This book is adapted for use in self-instruction." [Preface.] The eight chapters are entitled: Elementary Operations, Differentiation, Integration, Cylindrical and Spherical Coordinates, Electrostatic Fields, Applications to Current Electricity, Bipolar coordinates, The Betatron. Sixteen illustrative problems are listed and worked out in the text. An index is included.

Craig, Homer V. On extensors, first order partial differential equations and Poisson brackets. Tensor (N.S.) **6** (1956), 159–164.

The article contains two interesting applications of the

extensor construction to the partial differential equation

$$F[x^1, \cdots, x^N; y_1 \cdots, y_N; z(x^1, \cdots, x^N)]=0,$$

where $y_a=\partial z/\partial x^a$, and to the Poisson brackets. First, let $F(x; y; z)$ be a function of class c' of the $2N+1$ arguments $x^1, \cdots, x^N; y_1, \cdots, y_N; z$. We specialize F by taking z to be a class c' function of the variables x's, and the y's to be the components of a covariant vector. Moreover, let $\bar{F}[\bar{x}; \bar{y}; \bar{z}(\bar{x})]$ be the mate function obtained by replacing the x's in the symbol $F[x; y; z(x)]$ according to the coordinate transformation $x^a=x^a(\bar{x}^1, \cdots, \bar{x}^N)$ in a space S_N. Then it is shown that an attempt to construct extensor components by partial differentiation with respect to x's and y's of the function F leads to the extensor $E_{\alpha a}$:

$$E_{0a}=\frac{\partial F}{\partial x^a}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x^a},\ E_{1a}=-y_a,$$

if we confine ourselves to a class c' arc satisfying the equations $x'^a=\partial F/\partial y_a$ in S_N. On the other hand, the covariance of $-y_a$ implies that $T_{1a}=-y_a$, $T_{0a}=-y'_a$ are the components of an extensor $T_{\alpha a}$. Consequently, the equality of T and E leads to

$$x'^a=\frac{\partial F}{\partial y_a},\ y'_a=-\frac{\partial F}{\partial x^a}-\frac{\partial F}{\partial z}\frac{\partial z}{\partial x^a},\ z'=y_a\frac{\partial F}{\partial y_a},$$

which is precisely the set of differential equations for the characteristics [C. Caratheodory, Variationsrechnung, Leipzig und Berlin, 1935, p. 38]. If F does not contain z explicitly, namely $x'^a=\partial F/\partial y_a$, $y'_a=-\partial F/\partial x^a$, and if $G(x, y)$ and $x^a=x^a(t)$, $y_a=y_a(t)$ are all of class c', then we may write

$$G'=\frac{\partial G}{\partial x_a}x'^a+\frac{\partial G}{\partial y_a}y'_a=\Sigma\left(\frac{\partial G}{\partial x^a}\frac{\partial F}{\partial y_a}-\frac{\partial F}{\partial x^a}\frac{\partial G}{\partial y_a}\right),$$

which is nothing but the Poisson bracket $[G, F]$ of G and F. Next, let H be the total energy in mechanics; then we have available the well-known relationships:

$$p_a=g_{ab}x'^b=\partial H/\partial x'^a,\ x'^a=g^{ad}p_d.$$

We put $h(x, p)=H(x, g^{ab}p_b)$ and $f(x, p)=F(x, g^{ab}p_b)$. If $F(x, x')$ is as indicated a function of x's and x''s. Then the author finds out after some calculation

$$[f, h]=F_{;\alpha a}L_c^{\alpha a}g^{bc}H_{;1b}-H_{;\alpha a}L_c^{\alpha a}g^{bc}F_{;1b},$$

that is, the Poisson bracket is the difference between two invariants each of which is an extensor contraction involving primary extensors associated with F and H, the metric tensor and the metric extensor L. The notations employed there are given by H. V. Craig [Amer. J. Math. **59** (1937), 764–774; Vector and tensor analysis, McGraw-Hill, New York, 1943; Bull. Amer. Math. Soc. **53** (1947), 332–342; MR **5**, 77; **8**, 491] and A. Kawaguchi [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **9** (1940), 1–152; MR **2**, 22]. *A. Kawaguchi* (Sapporo).

See also: Mařík, p. 796; Aržanyh, p. 808.

Functions with Particular Properties

Lavrent'ev, M. M. On the inverse problem of potential theory. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 389–390. (Russian)

The inverse problem of potential theory, in a familiar formulation, is the following: Determine the form and position of a body B of given constant density, lying interior to a given closed surface Σ, when the value of the potential function due to B is given at each point of Σ. Under a suitable restriction on B the solution is known to be unique and stable, and an effective solution through approximations is possible.

The author now considers an alternative formulation in which both the potential and its normal derivative are given, but only on a portion of Σ. Uniqueness results are known; in the plane case the stability follows from a result of Carleman; and in this case an effective method of approximate solution has been indicated by the author [same Dokl. (N.S.) **102** (1955), 205–206; MR **17**, 146].

The following theorem giving the character of the stability of the solution of the 3-dimensional problem is announced and the author states that, with it as a basis, a solution through approximations can be effected for the class of bounded functions, for an arbitrary sufficiently smooth surface of Σ, and for an arbitrary portion of Σ bounded by a sufficiently smooth curve.

Let S be a bounded, simply connected plane domain, let T be its boundary, and let the function $\varphi(x, y)$ be continuous on $S+T$, vanish on T, and be positive and have continuous second-order partial derivatives on the interior of S. Denote by D the domain bounded by the surfaces $z=-\varphi(x, y)$ and $z=(h-1)\varphi(x, y)$ for some $h>0$. If the function $u(x, y, z)$ is harmonic in D and satisfies

$$\iint_S \operatorname{grad}^2 u[x, y, -\varphi(x, y)]dxdy<m^2,$$

$$\iint_S \operatorname{grad}^2 u[x, y, (h-1)\varphi(x, y)]dxdy<M^2,$$

then the inequality

$$\iint_S \operatorname{grad}^2 u[x, y, z\varphi(x, y)]dxdy<cM^{(z+1)/h}m^{2(h-z)/h},$$

where c is a constant independent of the function u, holds for $-1<z<h-1$. *E. F. Beckenbach.*

Friedman, Avner. Mean-values and polyharmonic polynomials. Michigan Math. J. **4** (1957), 67–74.

The author considers the problem of extending, to the case of polyharmonic polynomials, results of J. L. Walsh [Bull. Amer. Math. Soc. **42** (1936), 923–930] and of M. O. Reade and the reviewer [Trans. Amer. Math. Soc. **53** (1943), 230–238; MR **4**, 199] concerning the characterization of classes of harmonic polynomials $u(x, y)$ through properties of the mean values of the functions on the vertices, perimeters, and so on, of regular polygons.

Thus the following result is obtained (cf. the next review): Denote by $[(\alpha, \beta), n, R, \phi]$ a regular n-gon with center (α, β) and vertices

$$[\alpha+h\cos(\phi+2\pi k/n), \beta+h\sin(\phi+2\pi k/n)],$$

where $R=h\cos\pi/n$ and $k=1, \cdots, n$. Let $u(x, y)$ and its partial derivatives of the first $2p$ orders be continuous in a domain D, and for some $n>2p$ and all sufficiently small R let $u(x, y)$ satisfy the mean-value equation

$$(1)\ \frac{1}{sR}\iint_{[(\alpha,\beta),n,R,\phi]}u(x, y)dxdy= u(\alpha, \beta)+\sum_{j=1}^{p-1}\frac{\gamma_{j,n}R^{2j}\Delta^j u(\alpha, \beta)}{(2^j j!)^2(j+1)},$$

where s is the area of $[(0, 0), n, 1, 0]$ and the $\gamma_{j,n}$ are suitably defined constants. Then $u(x, y)$ is a p-harmonic polynomial of degree at most pn, and the pnth derivative of $u(x, y)$ in the ϕ direction vanishes.

The author shows, however, that there are p-harmonic polynomials of degree at most pn, for which the pnth

derivative in the ϕ direction vanishes, but for which the foregoing mean-value property (1) does not hold. Thus the characterization of the class of smooth functions for which (1) holds is here left incomplete.
E. F. Beckenbach (Los Angeles, Calif.).

Reade, Maxwell O. Remarks on a paper by A. Friedman. Michigan Math. J. 4 (1957), 75–76.

It is shown that the conclusion of the result stated in the preceding review still holds when the hypothesis is weakened (a) by removing the specification as to the particular values of the coefficients of the polynomial in the variable R of degree at most $2p-2$ on the right-hand side of the mean-value equation (1) [we note that (1) might equally well be replaced by a limiting relation] and (b) by assuming continuity only of the function $u(x, y)$ itself and not of its derivatives of the first $2p$ orders.
E. F. Beckenbach (Los Angeles, Calif.).

Brelot, Marcel. On the behavior of harmonic functions in the neighborhood of an irregular boundary point. J. Analyse Math. 4 (1955/56), 209–221.

Précisions relatives à un résultat antérieur de l'auteur [Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 22 (1946), 205–219; MR 8, 581] concernant l'existence d'une pseudo-limite λ en un point-frontière irrégulier Q d'un domaine euclidien Ω pour une fonction u harmonique dans Ω, bornée au voisinage de Q. Résultat central: si M_n est une suite maximale tendant vers Q (i.e. telle que $G_{P_0}(M_n)$ tende vers la pseudo-limite en Q de G_{P_0}, fonction de Green de Ω pour le pole arbitraire P_0), alors $u(M_n)$ tend vers λ. La question se pose de généraliser cet énoncé aux fonctions harmoniques positives et aux espaces de Green [M. Brelot et G. Choquet, Ann. Inst. Fourier, Grenoble 3 (1951), 199–263; MR 16, 34].
J. Deny.

Lisenkov, N. On mean values of subharmonic functions on spheres. Moskov. Gos. Univ. Uč. Zap. 145, Mat. 3 (1949), 108–115. (Russian)

The author shows that if the subharmonic function $U(\theta)$ has a finite value U_0 at θ_0, then $U(\theta)$ is asymptotically continuous at θ_0 in the following sense: For any region S_ρ inside the sphere with center θ_0 and radius ρ, bounded in part by the surface of the sphere and otherwise by planes passing through θ_0, and for any fixed positive ε, if M_ρ denotes the set of points of S_ρ for which the inequality $|U-U_0|<\varepsilon$ holds, then the measure $\mu(M_\rho)$ satisfies the relation $\lim_{\rho\to 0}[\mu(M_\rho)/\mu(S_\rho)]=1$. It might be noted that a simpler but equivalent formulation can be given.

Several applications are made. Thus it is shown that if the subharmonic function $U(\theta)$ takes on values a and b in its connected domain of definition D, then in D it takes on every value between a and b. Again, the mean value of U on the interior of any sphere in D must be taken on at some point in the sphere.
E. F. Beckenbach.

Friedman, Avner. Bilinear integrals of polyharmonic functions and of analytic functions. Michigan Math. J. 4 (1957), 77–84.

The Gustin bilinear integral identity for harmonic functions, and his applications of this identity [Amer. J. Math. 70 (1948), 212–220; MR 9, 352] are generalized in the present paper to an analogous identity and analogous results concerning polyharmonic functions.

For $i=1, 2$, let $f_i(q_i)$ be a real-valued n_i-harmonic function in an open set D_i of N-dimensional euclidean space ($N\geqq 2$); for a given point q_i of D_i, let R_i be such that the sphere Q_i with center q_i and radius R_i is contained, together with its interior, in D_i; and let x be a point on the unit sphere S. The author applies Gustin's result to show that, for $0\leqq r_i\leqq R_i$, we have an identity of the form

$$\int_S f_1(q_1+r_1x)f_2(q_2+r_2x)dS_x = \sum_{j=0}^{n_1-1}\sum_{k=0}^{n_2-1} A_{jk}(q_1, q_2, r_1r_2)r_1^{2j}r_2^{2k}.$$

The essential point is that the A_{jk} depend on the product r_1r_2 and not on the particular values r_1 and r_2 separately.

A bilinear integral characterization of n-harmonic functions, a theorem concerning the identical vanishing of polyharmonic functions, a generalization of Liouville's theorem, and other results are given as consequences of the foregoing identity.

The method is further extended to the analysis of analytic functions of a real variable. *E. F. Beckenbach.*

Erdélyi, Arthur. Singularities of generalized axially symmetric potentials. Comm. Pure Appl. Math. 9 (1956), 403–414.

On généralise un théorème de Szegö [J. Rational Mech. Anal. 3 (1954), 561–564; MR 16, 34] comparant les singularités d'une série de fonctions harmoniques zonales et d'une série entière associée: Considérons l'équation étudiée par Weinstein [Bull. Amer. Math. Soc. 59 (1953), 20–38; MR 14, 749]

$$(1)\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{k}{y}\frac{\partial u}{\partial y}=0,$$

où k est un nombre réel différent d'un entier négatif impair (pour $k=n-2$ c'est l'équation des fonctions harmoniques dans R^n, dépendant seulement de $x_1=x$ et de $(x_2+\cdots+x_n)^{\frac{1}{2}}=y$); à toute solution de (1) paire en y, régulière dans un domaine contenant un intervalle I_0 de l'axe Ox, on peut associer canoniquement un domaine de régularité maximum parmi ceux qui contiennent I_0 et dont l'intersection avec toute parallèle à Oy est un intervalle (éventuellement vide) symétrique par rapport à Ox; théorème: ce domaine de régularité ne dépend que des valeurs prises par u sur I_0; il est indépendant de k.
J. Deny (Strasbourg).

★ Szegö, G. Relations between different capacity concepts. Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 139–145. University of Maryland Book Store, College Park, Md., 1956.

Pólya et Szegö [Isoperimetric inequalities in mathematical physics, Princeton, 1951; MR 13, 270] ont conjecturé qu'entre les capacités newtonienne et logarithmique C et c d'une plaque D on a la relation $C/c\leqq 2/\pi$, autrement dit le maximum de C/c est atteint pour une plaque circulaire. Posons $k=\frac{1}{2}\pi \sup_D C/c$; sans parvenir à établir $k=1$, l'auteur indique diverses méthodes qui conduisent à $k<1{,}07$, ce qui améliore un peu le résultat antérieur $k<1{,}12$ [Szegö, Two applications of conformal mapping to the theory of Newtonian potential, Tech. Rep. no. 24, Stanford Univ., 1952].
J. Deny.

See also: Nikol'skiĭ, p. 795.

Special Functions

Maximon, Leonard C. A generating function for the product of two Legendre polynomials. Norske Vid. Selsk. Forh., Trondheim **29** (1956), 82–86 (1957).

The author evaluates the infinite series

$$S_1=\sum_{n=0}^{\infty} z^n P_n(\cos\alpha)P_n(\cos\beta)$$

in which P_n is the Legendre polynomial, $|z|<1$, and α and β are real. It is shown that the sum is

$$S_2=\left\{1-2z\cos(\alpha+\beta)+z^2\right\}^{-\frac{1}{2}} F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{4z\sin\alpha\sin\beta}{1-2z\cos(\alpha+\beta)+z^2}\right),$$

where F is the hypergeometric function. In particular, the author is interested in the sum of the series if $z=1$. The only singularities of S_2 on the unit circle are given by $z=\exp\{i(\pm\alpha\pm\beta)\}$. From this the author infers that S_1 converges at $z=1$ with the exception of $\alpha\pm\beta\equiv 0 \pmod{2\pi}$. He apparently overlooks the distinction between convergence and summability of the series S_1. For example, if $\alpha=0$ and $\beta=\pi$ the series S_1 diverges at $z=1$ although it is then summable by S_2. *C. J. Bouwkamp.*

Bhonsle, B. R.; and Varma, C. B. L. On some integrals involving Legendre function, associated Legendre function and Jacobi polynomials. Bull. Calcutta Math. Soc. **48** (1956), 103–108.

Sequences, Series, Summability

Hope-Jones, W. "A pretty series" (Notes 2419 and 2559). Math. Gaz. **41** (1957), 47–48.

The series in question is

$$(*) \qquad \sum_{1}^{\infty}\binom{3r+1}{r}\left(\frac{2}{27}\right)^r=\frac{1}{2}.$$

The result was proved by Watson [Math. Gaz. **39** (1955), 297] making use of the quadratic transformation of the hypergeometric series. The present paper contains an elementary proof due to G. H. Stevens. The coefficient of $1/y$ in the expansion of

$$E=\left\{1-z\left(\frac{1}{y}+y^2\right)\right\}^{-1} (z=x^{\frac{1}{3}})$$

is seen to be

$$\sum_{0}^{\infty}\binom{3r+1}{r}z^{3r+1}.$$

On the other hand for $x<4/27$, $y^3-y/z+1=0$ has three real roots $-\alpha, \beta, \gamma$, where α, β, γ are all positive, $\beta>\gamma$. Then expanding

$$E=\frac{1}{\alpha\beta z(1+y/\alpha)(1-y/\beta)(1-\gamma/y)},$$

it is found that the coefficient of $1/y$ is equal to

$$\frac{z^{-2}\gamma}{(\alpha+\gamma)(\beta-\gamma)}-1.$$

In particular, for $x=2/27$, this reduces to (*). *L. Carlitz* (Durham, N.C.).

Ionescu-Ţiu, C. On a method of summation. Gaz. Mat. Fiz. Ser. A. **8** (1956), 464–466. (Romanian)

Wynn, P. On a device for computing the $e_m(S_n)$ transformation. Math. Tables Aids Comput. **10** (1956), 91–96.

It is well-known that the convergence of slowly convergent (or even divergent) sequences can often be improved by Aitken's δ^2-process. Besides other things, R. J. Schmidt [Phil. Mag. (7) **32** (1941), 369–383; MR **3**, 276] gave a generalisation of Aitken's method which is closely related to the theory of continued fractions. Unfortunately Schmidt's formulae to compute a new sequence $\varepsilon_{2k}^{(n)}$ from a given sequence $\varepsilon_0^{(n)}$ $(n=0, 1, \cdots)$ require the evaluation of two k-row determinants for every n and are therefore too complicated for practical purposes, though the method as such would be very efficient. — In the paper reviewed here the author gives a method to compute the $\varepsilon_s^{(n)}$ recursive with respect to the index s: Starting with the original sequence $\varepsilon_0^{(n)}$ and introducing the auxiliary values $\varepsilon_{-1}^{(n)}\equiv 0$, we compute for $s=0, 1, 2, \cdots$ and for $n=0, 1, 2, \cdots$:

$$(1) \qquad \varepsilon_{s-1}^{(n+1)}+\frac{1}{\varepsilon_s^{(n+1)}-\varepsilon_s^{(n)}}=\varepsilon_{s+1}^{(n)}.$$

If further the ε's are arranged in a two-dimensional array like a difference table:

$$(2) \qquad \begin{array}{ccccc}
\varepsilon_0^{(0)} & & & & \\
 & \varepsilon_1^{(0)} & & & \\
\varepsilon_0^{(1)} & & \varepsilon_2^{(0)} & & \\
 & \varepsilon_1^{(1)} & & \varepsilon_2^{(0)} & \cdots \\
\varepsilon_0^{(2)} & & \varepsilon_2^{(1)} & & \\
 & \varepsilon_1^{(2)} & & & \\
\varepsilon_0^{(3)} & & \varepsilon_2^{(2)} & & \\
\vdots & \vdots & \vdots & &
\end{array}$$

then formula (1) allows to build up that table column by column in a very simple manner and the limit of the original sequence appears as limit of any of the even numbered columns $\varepsilon_{2k}^{(n)}$ $(n\to\infty)$, preferably for large k. In many cases the diagonal sequence $\varepsilon_{2k}^{(0)}$ $(k=0, 1, \cdots)$ converges fastest, since these values are approximants of a continued fraction. *E. Stiefel* (Zurich).

Vučković, Vladeta. Sur la construction des méthodes de limitation qui sont équivalentes et pas consistentes. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 89–96.

S. Mazur und W. Orlicz [Studia Math. **14** (1954), 129–160; MR **16**, 814] und L. Włodarski [ibid. **14** (1954), 161–187, 188–199; MR **16**, 814] gaben Beispiele permanenter Limitierungsverfahren, die nicht miteinander verträglich (consistent) sind. Der Verf. gibt daran anschliessend eine allgemeine Methode zur Konstruktion derartiger Beispiele. Es wird gezeigt: Ist $p_n\downarrow 0$, so ist notwendig und hinreichend dafür, dass eine Folge $\{u_n\}$ von der Form $\{\tau_n+c/p_n\}$ ist mit $\{\tau_n\}$ konvergent und $\tau_n-\tau_{n-1}=o((p_n-p_{n-1})/p_{n-1})$, $n\to\infty$, jede der Bedingungen (i) $U_n=(u_np_n-u_{n-1}p_{n-1})/(p_n-p_{n-1})$ ist konvergent (Grenzwert $\lim\tau_n$), (ii) $V_n=U_n+p_nu_n$ ist konvergent (Grenzwert $c+\lim\tau_n$). Ist $\limsup p_n/p_{n-1}<1$, so werden durch U_n und V_n zwei Limitierungsverfahren der gewünschten Form gegeben. *A. Peyerimhoff.*

Parameswaran, M. R. On the constants associated with a reversible summability matrix. Proc. Amer. Math. Soc. **8** (1957), 341–344.

The matrix $A=(a_{nk})$ is reversible if the equations

$$\sum_{k=0}^{\infty} a_{nk}x_k=y_n \quad (n=0, 1, 2, \cdots)$$

have a unique solution $\{x_k\}$ for each convergent sequence

$\{y_n\}$. It is known that the solution in this case is given by

$$x_k = c_k \lim_{n\to\infty} y_n + \sum_{n=0}^{\infty} b_{nk} y_n \quad (k=0, 1, 2, \cdots).$$

The author proves a number of results connecting the summability properties of A with the boundedness or unboundedness of the sequences $|c_k|$, $\sum_{n=0}^{\infty} |b_{n,k}|$.
H. G. Eggleston (Cambridge, England).

Ramanujan, M. S. Existence and classification of products of summability matrices. Proc. Indian Acad. Sci. Sect. A. **44** (1956), 171–184.

Der Verfasser betrachtet konvergenztreue und permanente Matrixtransformationen, die Folgen in Folgen, Folgen in Reihen, Reihen in Folgen und Reihen in Reihen überführen und untersucht die Eigenschaften der Produkte solcher Matrizen (z.B. führt das Produkt zweier Matrizen, die konvergente Folgen in konvergente Folgen überführen, wieder konvergente Folgen in konvergente Folgen über). Es ist nicht möglich, im Rahmen eines Referates auf die zahlreichen Ergebnisse näher einzugehen. Wir greifen noch das folgende Ergebnis heraus: Die Menge der Matrizen $P=(p_{nk})$, die konvergente Folgen $\{s_k\}$ durch $v_n=\sum_{k=0}^{\infty} p_{nk}s_k$ in konvergente Reihen $\sum v_n$ überführen, bilden mit der Norm

$$\|P\| = 2 \sup_n \sum_{k=0}^{\infty} \Big|\sum_{i=0}^{n} p_{ik}\Big|$$

eine Banachalgebra. *A. Peyerimhoff* (Giessen).

Gerrish, F. Conservative sequence-to-series transformation matrices. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 60–72.

The four methods of defining generalized limits or sums by infinite matrices, i.e., sequence-to-sequence, series-to-sequence, series-to-series, and sequence-to-series, are discussed in some detail in their relationships to each other. Products of the summability matrices used in these methods are investigated, and, in all cases when the product exists, its type is ascertained. Associativity of triple products of these summability matrices is discussed in the cases when the product always exists for both bracketings. Some simple inclusion relations between the various classes of matrices are obtained.

In a Note added in proof, the author states that J. D. Hill [Tôhoku Math. J. **45** (1939), 332–337] had defined sequence-to-series transformation matrices, so that the contents of § 2 of the present paper are not new. He also states that he had then seen the manuscript of a paper by M. S. Ramanujan [see the paper reviewed above] which overlaps somewhat with the present paper, but is on the whole complementary with it. *R. G. Cooke* (London).

Estrugo, Jose Antonio. On a device for increasing the rapidity of convergence of certain series. Gac. Mat., Madrid (1) **9** (1956), 136–147. (Spanish)

The author has apparently discovered (with insufficient hypotheses) the special case $k=2p+1$, $c_1=c_{2p+1}=\frac{1}{2}$, $c_{p+1}=-1$, and $c_j=0$ otherwise, for $1\leqq j\leqq 2p+1$, of the following theorem proved in, for example, K. Knopp, Theorie und Anwendung der unendlichen Reihen [4. Aufl., Springer, Berlin, 1947, p. 243; MR **10**, 446]: If k is fixed, $k\geqq 2$, $\lim x_n=\xi$ and $a_n=\sum_{i=1}^{k} c_i x_{n+i}$ for $n=0, 1, 2, \cdots$, then $\sum_0^{\infty} a_n$ is convergent to

$$\sum_{j=1}^{k-1} x_j \sum_{i=1}^{j} c_i + \xi \sum_{i=2}^{k-1} (i-1)c_i.$$

He then applies this result to many special sequences $\{x_n\}_1^{\infty}$. In particular, if p is odd, he finds that $\sum_1^{\infty} x_n$ can some times be converted into a more rapidly convergent series by the above device. *A. E. Livingston.*

See also: Wright, p. 793; Duffin, p. 804; Zagustin, p. 828; Aržanyh, p. 840; Ionescu, p. 842; Ionescu, p. 842; Epstein, p. 848; McCarthy, p. 851.

Approximations, Orthogonal Functions

Pol'skiĭ, N. I. On a general scheme of application of approximation methods. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 1181–1184. (Russian)

The approximation methods of Ritz, Galerkin, N. M. Krylov, and others are subsumed in a "generalized" result which has its fair share of restrictions. The distillation of their work takes a Banach space form as follows: Let K be a linear operator from E_1 onto E_2. Assume that $\{\psi_i\}$ is a countable set in E_1 such that the set $\{K\psi_i\}$ is linearly independent and linearly dense in E_2 $(=KE_1)$. Let $\{\varphi_i\}$ be linearly independent and linearly dense in E_2 and let M_n be the space spanned by $\varphi_1, \cdots, \varphi_n$ and L_n the space spanned by $K\psi_1, \cdots, K\psi_n$. Let P_n be a projection on M_n. If for some n_0, there is a constant C for which $\|y\|\leqq C\|P_n y\|$, $y\in L_n$, then solutions X_n of the equations $P_nKx_n=P_nf$ converge to a solution of $Kx=f$.
B. Gelbaum (Minneapolis, Minn.).

Rangaswami Aiyar, K. On a problem in curve fitting. J. Annamalai Univ. Part B. **20** (1956), 80–82.

The author calculates the Fourier-Hermite coefficients for a function whose graph has the shape of an isosceles trapezoid, and illustrates the 2-term approximation graphically. *R. P. Boas, Jr.* (Evanston, Ill.).

Ahiezer, N. I. On weighted approximations of continuous functions by polynomials on the entire number axis. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 4(70), 3–43. (Russian)

In this survey paper the author unifies a number of results including recent work on the problem described in the title, states the relevant theorems and presents proofs. His references include work of S. Bernstein, Carleson, Izumi and Kawata, Mandelbrojt, Carleman, Ostrowski, Hall, Mergelyan [see also MR **18**, 734], Levin, Pollard, Vidav and M. Riesz.
N. Levinson (Cambridge, Mass.).

Butzer, Paul L. On the singular integral of de la Vallée-Poussin. Arch. Math. **7** (1956), 295–309.

For real-valued, periodic functions $f(x)$ (with period 2π) the singular integral is considered

$$V_n(x) = \int_{-\pi}^{+\pi} K_n(x, t) f(t) dt,$$

where the kernel $K_n(x, t) = h_n \cos^{2n}(\frac{1}{2}(t-x))$ with $h_n^{-1} = \int_{-\pi}^{+\pi} \cos^{2n} u\, du$. This $V_n(x)$ is a trigonometric polynomial of order n at most. Let $C_{2\pi}$ denote the class of the continuous periodic functions $f(x)$. First $f(x)\in C_{2\pi}$ is assumed; then it is well known that for all real x one has uniformly $\lim_{n\to\infty} V_n(x) = f)x)$. The author investigates the degree of approximation under suitable assumptions, generalizing and complementing results of I. P. Natanson [Constructive theory of functions, Gostehizdat, Moscow-Lenin-

grad, 1949, Ch. 10, § 3; MR 11, 591; 16, 1100]. The functions

$$\omega(\delta)=\sup_{0<|h|\leq\delta} |f(x+h)-f(x)|,$$

$$\omega^*(\delta)=\sup_{0<|h|\leq\delta} |f(x+h)+f(x-h)-2f(x)|$$

are called modulus of continuity and generalized modulus of continuity of f, respectively. The function f satisfies a Lipschitz condition of order α or $f\in\mathrm{Lip}\,\alpha$ if $\omega(\delta)\leq M\delta^\alpha$, $0<\alpha\leq 1$, M a constant. If $\omega(\delta)=o(\delta^\alpha)$, the author writes $f\in\mathrm{Lip}^*\,\alpha$. Then the proves he following theorems: If $f\in C_{2\pi}$ has a generalized modulus of continuity $\omega^*(\delta)$, then, for all x, $|V_n(x)-f(x)|\leq\frac{3}{2}\omega^*(n^{-\frac{1}{2}})$. If $f\in C_{2\pi}$ and if f' exists at x, then for this x one has

$$\lim_{n\to\infty} n^{\frac{1}{2}}|V_n(x)-f(x)|=0.$$

If f has a derivative $f'\in C_{2\pi}$ with modulus of continuity $\omega_1(\delta)$, then for all x one has $|V_n(x)-f(x)|\leq Cn^{-\frac{1}{2}}\omega_1(n^{-\frac{1}{2}})$.

Now f is assumed L-integrable (instead of continuous). The following theorem is proved by the author: If $f\in L_1$, then, for almost every x, $\lim_{n\to\infty} V_n(x)=f(x)$. If the kth derivative $f^{(k)}(x)\in L_1$ and $f^{(k-1)}(x)$ is absolutely continuous, then $\lim_{n\to\infty} V_n^{(k)}(x)=f^{(k)}(x)$ a.e.

If $f(x)$ is periodic (with period 2π) and $f\in L_p$, $p\geq 1$, with

$$\|f\|_p=\left[\int_{-\pi}^{+\pi} |f(x)|^p dx\right]^{1/p},$$

the "integral modulus of continuity of f" is defined by the author as $\omega_p(\delta)=\sup_{0<|h|\leq\delta} \|f(x+h)-f(x)\|_p$ and the "generalized integral modulus of continuity of f" as $\omega_p^*(\delta)=\sup_{0<|h|\leq\delta} \|f(x+h)+f(x-h)-2f(x)\|_p$. If $\omega_p(\delta)\leq M\delta^\alpha$ $(0<\alpha\leq 1)$, then he says that f satisfies an "integral Lipschitz condition" or $f\in\mathrm{Lip}(\alpha, p)$. If $\omega_p(\delta)=o(\delta^\alpha)$, he writes $f\in\mathrm{Lip}^*\,(\alpha, p)$. Then he proves the following theorems concerning approximation in the mean: If f has a generalized integral modulus of continuity $\omega_p^*(\delta)$, $p\geq 1$, then

$$\|V_n-f\|_p=O[\omega_p^*(n^{-\frac{1}{2}})].$$

If $f'\in\mathrm{Lip}\,(\alpha, p)$ $(p\geq 1;\ 0<\alpha\leq 1)$, then

$$\|V_n-f\|_p=O(n^{-\frac{1}{2}(\alpha+1)}).$$

Finally the author discusses some applications, proving the following theorems: If $f\in C_{2\pi}$, then a necessary and sufficient condition for $f^{(k)}$ to exist and to belong to Lip 1 is that $|V_n^{(k+1)}(x)|=O(1)$ uniformly in n and x. Let $f^{(k-1)}$ be absolutely continuous and $f^{(k)}\in L_1$; then $f^{(k)}\in\mathrm{Lip}\,(\alpha, p)$ $(p\geq 1;\ 0<\alpha\leq 1)$ if and only if

$$V_n^{(k)}\in\mathrm{Lip}(\alpha, p)$$

uniformly in n. Let $f\in L_1$; then f is almost everywhere equal to a function of bounded variation if and only if $\|V_n'\|_1=O(1)$. *A. Rosenthal* (Lafayette, Ind.).

Potapov, M. K. On Jackson type theorems in the L_p metric. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 1185–1188. (Russian)

The theorems of Jackson about the approximation of functions $f(x)$ defined on $(-1, +1)$ by algebraic polynomials are extended here to the metric L^p. Typical is the following theorem. Let $H_p^{r+\alpha}(N)$ be the class of functions $\varphi(x)$, $-1\leq x\leq +1$, having an rth derivative $\varphi^{(r)}\in L^p$ and satisfying a condition

$$\left\{\int_c^d |\varphi(x+h)-\varphi(x)|^p dx\right\}^{1/p}\leq M|h|^\alpha \quad (0<\alpha\leq 1)$$

for each interval $(c, d)\subset(-1, +1)$ such that $x\in(c, d)$ implies $x+h\in(-1, +1)$, where M is independent of c, d, h. Then if $f(x)\in H_p^{r+\alpha}(M)$, $0<\alpha\leq 1$, and $n\geq r+1$, there is a polynomial P_n of degree $\leq n$ and such that

$$\int_{-1}^{+1}\left|\frac{f(x)-P_n(x)}{\{(1-x^2)^{\frac{1}{2}}+n^{-r}\}^r}\right|^p dx\leq\frac{cM}{n^{r+\alpha}}.$$

A. Zygmund (Chicago, Ill.).

Ul'yanov, P. L. On unconditional convergence almost everywhere. Mat. Sb. N.S. **40(82)** (1956), 95–100. (Russian)

If $\{P_n\}_0^\infty$ is the ON system of polynomials associated with the positive weight function w on the finite interval $[a, b]$ and if $fw\in L[a, b]$, set $c_n(f)=\int_a^b fwP_ndx$. If, now, $\phi(t)$ is positive and non-decreasing for $t\geq m$,

$$\sum_m^\infty[\phi(n)n\ln n]^{-1}<\infty,\ \phi(n)\ln^2 n=O\{\sum_m^n\phi(k)\ln k/k\} \text{ for } n>m,$$

and if $\{n_k\}_1^\infty$ is an increasing sequence of natural numbers for which $\ln n_{k+1}=O(\ln n_k)$ and $\sum_1^\infty 1/\phi(n_k)<\infty$, then, corresponding to each rearrangement of the Fourier series $\sum c_n(f)P_n(x)$, there is a subset of $[a, b]$ of measure zero off which the rearranged series converges to $f(x)$. The author states and proves this result for $\phi(t)=(\ln\ln t)^{1+\epsilon}$, $\epsilon>0$ (and $n_k=2^{2^k}$) by showing that $\sum_m^\infty |c_n(f)|^2\phi(n)\ln^2 n<\infty$ and then appealing to a result of Orlicz [see, e.g., S. Kaczmarz and H. Steinhaus, Theorie der Orthogonalreihen, Warszawa-Lwów, 1935, p. 170].

He shows also that if $f\in\mathrm{BV}[a, b]$ and if $w(x)$, in addition to being positive, is $O[(b-x)(x-a)]^{-\frac{1}{2}}$ on $[a, b]$, then $\sum |c_n(f)|^{1+\epsilon}$ and $\sum |c_n(f)|^2 n^{1-\epsilon}$ are convergent for every $\epsilon>0$. *A. E. Livingston* (Seattle, Wash.).

Trigonometric Series and Integrals

Postnikov, A. G. Estimation of an exponential trigonometric sum. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 661–666. (Russian)

Let g be an integer greater than unity and let

$$S=\sum_{x=0}^{P-1} e^{2\pi i\alpha g^x}$$

where $0\leq\alpha<1$. It is shown that this interval can be divided into two subsets $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that, for large P,

$$|S|\leq K(\epsilon)\frac{P}{\log^{\frac{1}{2}-\epsilon}P}$$

when $\alpha\in\mathfrak{M}_2$, and

$$\text{meas }\mathfrak{M}_1=O\{\exp(-K\log^3 P+O(\log P))\}.$$

Here K is a positive constant and $K(\epsilon)$ depends only on ϵ, which is positive.

The proof uses an upper bound for $|S|^{2d}$ obtained by the author in a previous paper [Dokl. Akad. Nauk SSSR (N.S.) **86** (1952), 473–476; MR **14**, 359] together with a lemma which states that the number of numbers of f digits in the scale of r in which a given digit b occurs more than $\eta f/r$ times, where $r>\eta>2$, is $O\{f^{\frac{1}{2}}r^f\exp(-\frac{1}{3}f\eta^2/r^2)\}$.
R. A. Rankin (Glasgow).

Ivašev-Musatov, O. S. On trigonometric null-series. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 438–441. (Russian)

In a previous paper [same Dokl. (N.S.) **82** (1952),

9-11; MR 14, 163] the author showed that given any function $\chi(u)$, $u \geqq 0$, non-negative, tending to 0 sufficiently regularly as $u \to +\infty$, and such that $\sum \chi^2(n) = \infty$, there is a Fourier-Stieltjes series $\sum \alpha_n e^{inx}$ of a singular and non-negative mass distribution F such that

$$(*) \qquad \alpha_n = (2\pi)^{-1} \int_0^{2\pi} e^{-inx} dF(x) = o(\chi(|n|)).$$

In particular for $\chi(u)$, and u large enough, he could take any of the functions

$$u^{-\frac{1}{2}},\ (u \log u)^{-\frac{1}{2}},\ (u \log u \log\log u)^{-\frac{1}{2}},\ \cdots.$$

The function F satisfying (*) had a spectrum everywhere dense. In the present note, modifying the previous construction, he obtains an F which has all the previous properties except that now the spectrum of F is a perfect non-dense set of measure 0, so that the resulting series $\sum \alpha_n e^{inx}$ satisfies (*) and converges to 0 almost everywhere. A corresponding result holds for Fourier-Stieltjes integrals. *A. Zygmund* (Chicago, Ill.).

Matsumoto, Kishi. Lebesgue's constant of (R, λ, k) summation. Proc. Japan Acad. 32 (1956), 658–661.

Für das Rieszsche Verfahren $(R, \lambda(\omega), k)$ mit $\lambda(\omega) = e^{\mu(\omega)}$ und differenzierbarem $\mu(\omega)$ $(0 \leqq \omega < \infty)$ berechnet Verf. die Lebesgueschen Konstanten zu $L(\omega) \cong 4\pi^{-2} \log \omega\mu'(\omega)$, falls $\lambda(\omega)$ die folgenden Eigenschaften besitzt: $\mu(\omega) \uparrow \infty$, $\mu'(\omega) \downarrow 0$ $(\omega > A)$, $\omega\mu'(\omega) \to \infty$, $\lambda'(\omega)$ wächst monoton für $\omega > A$. *A. Peyerimhoff* (Giessen).

Mohanty, R.; and Mohapatra, S. On the absolute convergence of a series associated with a Fourier series. Proc. Amer. Math. Soc. 7 (1956), 1049–1053.

Es sei

$$\frac{a_0}{2} + \sum_1^\infty (a_n \cos nx + b_n \sin nx) = \sum A_n$$

die Fourierreihe einer Funktion $f(t) \in L(-\pi, +\pi)$ mit den Teilsummen $s_n = \sum_{k=0}^n A_k$. Wird mit $\phi_\alpha(t)$ das (C, α)-Mittel der Funktion $\phi(t) = \frac{1}{2}(f(x+t) + f(x-t) - 2s)$ bezeichnet, so gilt: (1) Ist $\phi_1(t) \log (k/t)$ in $(0, \pi)$ von beschränkter Schwankung, $\phi_1(t)/t \in L(0, \pi)$ und $\{n^\delta A_n\}$ von beschränkter Schwankung für ein $\delta > 0$, so ist

$$(*) \qquad \sum \left| \frac{s_n - s}{n} \right| < \infty;$$

(2) Gilt (*), so ist $\phi_{1+\delta}(t)/t \in L(0, \pi)$ für jedes $\delta > 0$; (3) Ist $\phi(t)/t \in L(0, \pi)$, so ist $\sum (s_n - s)/n$ für jedes $\delta > 0$ $|C, \delta|$-summierbar. Zum Beweis von (1) wird aus den Voraussetzungen über $\phi_1(t)$ durch direkte Rechnung die $|R, e^{n^\alpha}, 1|$-Summierbarkeit $(0 < \alpha < 1)$ von $\sum (s_n - s)/n$ abgeleitet; aus der Bedingung über A_n folgt dann die Behauptung durch einen absoluten Taubersatz. Die Beweise für (2) und (3) ergeben sich unmittelbar aus entsprechenden Beweisen bei Bosanquet [Proc. London Math. Soc. (2) 41 (1936), 517–528; J. London Math. Soc. 11 (1936), 11–15]. *A. Peyerimhoff* (Giessen).

Integral Transforms

Duffin, R. J. Two-dimensional Hilbert transforms. Proc. Amer. Math. Soc. 8 (1957), 239–245.

The author mentions that the following six topics have a natural relationship: Fourier transforms, Abel summability, Poisson's integral, Cauchy's integral, conjugate harmonic functions, and Hilbert transforms. In the present paper he brings out similar relationships when the dimension is increased by one in each case. He first gives an evaluation of the Fourier transform (of a function of two variables) by the method of Abel summability and shows that this is consistent with the convergence in mean evaluation. This work leads, via the Poisson integral formula, to harmonic functions of three variables and to a relationship between Fourier transforms and these harmonic functions. The author then gives two alternative definitions of Hilbert transforms, discusses Abel summability for Hilbert transforms, and shows that a three-dimensional generalization of the Cauchy integral formula enables one to derive the Hilbert transform formulas. This latter work leads to quaternion functions and to an analog of Cauchy's theorem in three dimensions. *W. T. Martin*

Bhonsle, B. R. On two theorems of operational calculus. Bull. Calcutta Math. Soc. 48 (1956), 95–102.

Jaiswal, J. P. On Meijer transform. Ganita 6 (1955), 75–91 (1956).

The author uses known functional relations (generating functions, finite sums, and the like) for the Laguerre polynomial in order to obtain the corresponding relations for $\int_0^\infty e^{-st} L_n^{(\alpha)}(st) f(t) dt$. There are seven general formulas with examples to each formula. *A. Erdélyi.*

See also: Carleson, p. 798; Hačatryan, p. 798; Stelson, p. 858.

Ordinary Differential Equations

Kondyurin, Yu. N. On the convergence of the method of Čaplygin for the solution of a two-point boundary problem. Vestnik Leningrad. Univ. 11 (1956), no. 19, 73–79. (Russian)

The problem is $y'' = f(x, y, y')$; $y(0) = y(1) = 0$, where (1) f is twice differentiable with respect to y and y', and $f_{y'} > 0$ for $0 \leqq x \leqq 1$ and all y and y'; (2) for some constants A and B, and arbitrary y and y', $|f| \leqq A + By'^2$; (3) the form with $f_{yy}, f_{yy'}, f_{y'y'}$ as coefficients is semidefinite. Then with $y_0 \equiv 0$, the sequence $\{y_n\}$ is defined by $\eta_n = y_{n+1} - y_n$, $\eta_n(0) = \eta_n(1) = 0$, $\eta_n'' = f_y \eta_n + f_{y'} \eta_n' + f - y_n''$, the arguments being x, y_n and y_n'. The sequence so defined is shown to converge to the solution y.

A. S. Householder (Oak Ridge, Tenn.).

Kowalski, M. On the determinants of Wroński in linear rings. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 789–792 (1957).

Let C be a commutative ring with unit, K a commutative algebra over C, and D a fixed C-endomorphism of the C-module K. For $x_1, \cdots, x_m \in K$ define the "wronskian" $W(x_1, \cdots, x_m)$ to be $\det(D^{j-1}x_i)$. Theorem 1 asserts that if there exists a monic polynomial $f(\lambda) \in C[\lambda]$ of degree m and such that $f(D)$ annihilates $x_1, \cdots, x_m$, and if similarly there exists monic $g(D)$ of degree n annihilating $y_1, \cdots, y_n$, then

$$W(x_1, \cdots, x_m, y_1, \cdots, y_n) = W(x_1, \cdots, x_m) W(y_1, \cdots, y_n) R(g, f),$$

where R is the resultant of g and f. {Notice that if $x_1, \cdots, x_m$ are not annihilated by a polynomial of degree $< m$, then

f is unique. And if they are so annihilated, then

$$W(x_1, \cdots, x_m)=0.$$

Hence the factor $R(g, f)$ is ambiguous only when both sides of the equality vanish. The reviewer is indebted to A. Rosenberg for this remark.} Theorems 3 and 4 exhibit formulas for the wronskians of other special families $z_1, \cdots, z_{m+n}$ of solutions to $f(D)g(D)z=0$.

No proofs are given in the paper and the author never states clearly what restrictions he imposes on K. Theorem 2, for instance, is false unless K is essentially a vector space. And the first sentence in paragraph 6 is false unless the elements of C itself can be decomposed into irreducible factors. But presumably all is well in the case that really interests the author, where C is the real or complex numbers, K is the algebra of infinitely-differentiable functions, and D is a differential operator. *H. Mirkil.*

De Castro Brzezicki, A. Oscillating integrals of linear differential equations with second member. Las Ciencias 22 (1957), 5–28. (Spanish)

The classical qualitative integration methods of Sturm for the ordinary differential equations

$$\frac{d}{dx}[P(x)y']-Q(x)y=0 \text{ and } y''+p(x)y'+q(x)y=0$$

are concerned with the distribution of the zeros and the oscillating character of the solutions. The author extends many of these known properties to the corresponding non-homogeneous differential equations. Several interesting examples are given. *J. B. Diaz.*

Cuming, H. G. Perturbations of a body in an exponential atmosphere. Aircraft Engrg. 29 (1957), 123–124.

This note presents an analytic solution to the second order linear differential equation whose coefficients vary exponentially with time. The equation arises in the investigation of small angular perturbations about a linear path of a body traversing an atmosphere in which the density varies exponentially with height. The solution obtained is valid provided that the angular motion is lightly damped, a condition usually satisfied in practical cases. *Author's summary.*

Wintner, Aurel. Über eine Abschätzung der Amplituden in freien Schwingungsproblemen veränderlicher Kreisfrequenz. Z. Angew. Math. Phys. 7 (1956), 350–352.

Suppose that $x''+\omega^2(t)x=0$ has a solution $x(t)$ such that $x(0)=x(h)=0$, $x(t)>0$, for $0<t<h$; let $m=\max\{x(t):0\leq t\leq h\}$, $f=\int_0^h x(t)dt$, $\mu_n=\int_0^h t^n\omega^2(t)dt$ ($n=0, 1, 2$). Then (i) $\mu_1>1$ and $m/f<2(\mu_1-1)/\mu_2$, (ii) $m/f<4/\mu_0$. Result (i) is a simple deduction from the inequality $h<\int_0^h t(h-t)\omega^2(t)dt$, due to Hartman and Wintner [Amer. Math. 73 (1951), 885–890; MR 13, 652]. *G. E. H. Reuter.*

Vasilache, S. Sur la détermination d'un système fondamental de solutions d'une équation différentielle linéaire d'ordre n. Ann. Polon. Math. 3 (1956), 172–182.

Consider the ordinary differential equation

$$(*)\quad y^{(n)}(x)=a_1(x)y^{n-1}(x)+a_2(x)y^{(n-2)}(x)+\cdots+a_n(x)y(x)+f(x)\equiv L_x(y(x))+f(x).$$

The author points out that if $G(x,s)$ is a one-parameter family of solutions of the homogeneous equation corresponding to (*), defined by the conditions $\partial^n G(x, s)/\partial x^n=L_x(G(x, s))$, $\partial^i G(s, s)/\partial x^i=0$ ($i=0, 1, \cdots, n-2$), $\partial^{n-1}G(s, s)/\partial x^{n-1}=1$, then the general solution of (*) is $\sum_{i=1}^n c_iY_i(x)+Z(x)$, where $Z(x)\equiv\int_{x_0}^x G(x, s)f(s)ds$ which satisfies $Z^{(k)}(x_0)=0$, $k=0,\cdots,n-1$) is a particular solution of (*); while $Y_1(x),\cdots,Y_n(x)$ is a fundamental system of solutions of the homogeneous equation corresponding to (*) (which satisfies the Cauchy conditions $Y_i^{(j-1)}(x_0)=\delta_{ij}$, for $i, j=1, \cdots, n$; δ_{ij} being Kronecker's delta) given explicitly in terms of G as follows:

$$Y_i(x)=\frac{(x-x_0)^{i-1}}{(i-1)!}+\int_{x_0}^x G(x, s)L_s\frac{(s-x_0)^{i-1}}{(i-1)!}ds\ (i=1,\cdots,n).$$

For the direct determination of $G(x, s)$, the integro-differential equation

$$G(x, s)=\frac{(x-s)^{n-1}}{(n-1)!}+\int_s^x\frac{(x-t)^{n-1}}{(n-1)!}L_t(G(t, s))dt$$

is suggested. {Remark: Note the special case $L\equiv0$.} *J. B. Diaz* (College Park, Md.).

Miu, I. Integration of differential equations with constant coefficients with the aid of first integrals. Gaz. Mat. Fiz. Ser. A. 8 (1956), 467–471. (Romanian)

Drîmbă, C. Solution of systems of linear homogeneous differential equations with constant coefficients in the case of multiple roots of characteristic equations. Gaz. Mat. Fiz. Ser. A. 8 (1956), 459–463. (Romanian)

Zolotarev, Yu. G. On stability in the first approximation. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. 1956, no. 5(9), 62–70. (Russian)

The author considers the stability of the null solution of the nonlinear system $dx/dt=A(t)x+f(x, t)$ under a set of special assumptions concerning the nonlinear term $f(x, t)$ and the solutions of the linear system $dy/dt=A(t)y$. *R. Bellman* (Santa Monica, Calif.).

Garstens, Martin A. Noise in nonlinear oscillators. J. Appl. Phys. 28 (1957), 352–356.

A method is presented for estimating the nonlinear noise contribution in an oscillator at low levels of oscillation. This involves obtaining an approximate solution of a nonhomogeneous van der Pol type of nonlinear differential equation driven by noise. The method of solution is based upon the reduction of the equation to a linear form. The nonlinear contribution can then be obtained by computing the power spectrum output of a cube law rectifier. The calculations indicate a variation of nonlinear noise with rf level. Such variation is observed in nuclear and electronic magnetic resonance devices used for the detection of magnetic absorption in paramagnetic materials. *Author's summary.*

Gubar', N. A. Characterization of compound singular points of two differential equations by means of rough singular points of closely related systems. Mat. Sb. N.S. 40(82) (1956), 23–56. (Russian)

The author studies the real analytical systems in two variables

$$(1)\quad \dot{x}=P(x, y),\quad \dot{y}=Q(x, y)$$

which are reducible, by a linear transformation of coordinates and suitable choice of time to one of the two types

$$(a)\quad \dot{x}=X(x, y),\quad \dot{y}=y+Y(x, y),$$

$$(b)\quad \dot{x}=y+X(x, y),\quad \dot{y}=Y(x, y),$$

where X, Y are power series in x, y beginning with terms of degree at least two. His concern is the local phase portrait at the origin. The basic method utilized is to vary slightly P and Q in (1) to obtain

$$\dot{x}=P+p,\quad \dot{y}=Q+q, \tag{2}$$

where p, q together with their partials of order $\leq r$ are $<\varepsilon$ in absolute value near the origin. There appear then a certain number of ordinary singularities: nodes, foci, saddle points (these are the "rough" singularities) near the origin. The detailed classifications are obtained from an examination of the behavior of these ordinary singularities. The treatment is wholly analytical and altogether too complicated for a detailed description. It may be said that systems of type (a) have been discussed in detail by Bendixson [Acta Math. **24** (1901), 1–88] and those of type (b) more recently by Andreev [Vestnik Leningrad. Univ. **10** (1955), 43–65; no. 8, MR **17**, 364]. There is also on type (b) a recent Note by Barocio [Ann. of Math. Studies 36, Contributions to the theory of nonlinear oscillations, III. 127–135, 1956]. [Additional references: N. B. Haimov, Uč. Zap. Stalinabadsk. Inst., 3–30, 1952.]

S. Lefschetz (Mexico, D.F.).

See also: Wintner, p. 798; McKiernan, p. 807; Haplanov, p. 810; Faure, p. 835; Boltyanskiĭ, p. 859; Rumyancev, p. 859.

Partial Differential Equations

★ **Fantappiè, Luigi. Les nouvelles méthodes d'intégration, en termes finis, des équations aux dérivées partielles.** Second colloque sur les équations aux dérivées partielles, Bruxelles, 1954, pp. 95–128. Georges Thone, Liège; Masson & Cie, Paris, 1955.

Expository article of five different methods, developed by the author, for finding explicit solutions of partial differential equations. The basis of these methods is a functional calculus for operators, in the domain of analytic functions. *P. D. Lax* (New York, N.Y.).

Bouligand, G. Sur une classe d'équations $z=f(x, y, p, q)$. J. Math. Pures Appl. (9) **36** (1957), 65–66.

If the integral surfaces $z=z(x, y)$ of the first order partial differential equation $z=f(x, y, p, q)$ have the property that the characteristics of the equation are asymptotic curves, then

$$f_p(f_x-p)+f_q(f_y-q)=0. \tag{C}$$

Geometrically, this means that the integral surfaces $z=z(x, y)$ can be generated by characteristics and must possess curvatures of opposite sign at each point. The author shows that, in spite of its qualitative aspect, the fact that the curvatures have opposite signs at each point implies the relation (C) in the classical case of sufficiently differentiable surfaces, and raises the question of finding a more general (constructive) definition of "surfaces having curvatures of opposite sign" [cf. G. Bouligand, Rend. Sem. Mat. Univ. Padova **24** (1955), 53–69, pp. 63–65; MR **16**, 988]. *J. B. Diaz* (College Park, Md.).

Sonner, Hans. Über die Zurückführung partieller Differentialgleichungen auf gewöhnliche. Math. Z. **65** (1956), 483–493.

Le problème qui apparaît posé par le titre du présent mémoire est sans doute très nouveau dans la théorie des équations aux dérivées partielles, telle comme nous la connaissons depuis l'oeuvre des plus grands mathématiciens pendant deux siècles; non plus paraît-il possible d'en chercher la solution dans l'oeuvre récente de Bourbaki cité par l'A., celle-ci étant essentiellement dirigée à une analyse critique plutôt qu'à la résolution de problèmes.

Par autre côté une difficulté se présente tout de suite si seulement nous observons que la solution d'une équation differentielle ordinaire dépend en tous les cas d'un nombre fini de constantes arbitraires, tandis qu'une équation ou système aux dérivées partielles suppose l'appui complémentaire de variétés continues convenablement données dans l'espace ambiant. Et finalement le peu de calculs et de formules que nous offre la lecture du texte semble se limiter à l'usage de dérivées et d'intégrales du premier ordre, guère conformément à la généralité qui apparaît dans le titre. *B. Levi.*

Hartman, Philip. On Jacobi brackets. Amer. J. Math. **79** (1957), 187–189.

Let $z=z(x_1, \cdots, x_n)$ be the solution of two first order nonlinear partial differential equations $F=0$ and $G=0$. According to a standard theorem, z is then a solution of $[F,G]=0$, where $[F, G]$ denotes the Poisson bracket of F and G, provided that z is twice differentiable. In this note this result is proved for once differentiable z, on the basis of the following important theorem of Pliś [Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 419–422; MR **16**, 704]: Every once differentiable solution z of $F=0$ is generated by characteristic strips if the number of space variables is two. For more variables, a more restrictive result holds. *P. D. Lax* (New York, N.Y.).

Ostrowski, Alexander. Zur Theorie der partiellen Differentialgleichungen erster Ordnung. Math. Z. **66** (1956), 70–87.

The usual theory of characteristics of a first order partial differential equation (1) $F(z, p, q, x, y)=0$ is founded on the following. (a) If $z=\phi(x, y)$ is an integral surface touched by a characteristic element E_0 at $x_0, \cdots, q_0$, then the whole characteristic strip through E_0 touches the integral surface. (b) If C_0 is a datum strip $x_0(\tau), \cdots, q_0(\tau)$, then an integral surface passes through C_0. The present paper establishes (a) and (b) under Lipschitz conditions (modified by a 'slowly varying' monotone function as multiplier) rather than the differentiability hypotheses customarily imposed on F, ϕ and their partials.

D. G. Bourgin (Urbana, Ill.).

Nakamori, Kanzi. On a nonlinear boundary problem of a partial differential equation of elliptic type. Yokohama Math. J. **2** (1954), 165–172 (1955). (Esperanto)

The author gives an existence and uniqueness theorem (under suitable hypotheses) concerning the determination of a solution $u(x, y)$ of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}+cu=f$$

in a plane open set T minus a certain point P of T, near which $u(x, y)$ behaves like $\log(1/r)$ plus a regular function, r being the Euclidean distance from P; and which satisfies the boundary condition

$$\frac{\partial u}{\partial n}=\varphi(x, y, u),$$

on the boundary S of T, where φ is a given function of three variables. *J. B. Diaz* (College Park, Md.).

Halilov, Z. I. Approximation of solutions of boundary problems for general elliptic systems. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **6** (1953), 88–96. (Russian. Azerbaijani summary)

The approximation method described is based upon fulfilling exactly the differential equations of the given boundary value problem, while the boundary conditions are satisfied only approximately. It may best be outlined by the example of a boundary value problem consisting of a linear homogeneous equation $L(U)=0$ in a domain D, the boundary condition being of the form $R(U)=f$ on the boundary S of D, where R is a given linear operator and f is a given function. It is assumed that there is a relation $U=A(\varphi)$ which gives a one-to-one representation of the class of solutions of the differential equation $L(U)=0$ in terms of a linear class of functions φ; that the boundary value problem in question possesses a unique solution; and that the class of functions φ consists of all linear combinations, and limits of such linear combinations, of a fixed sequence of (known) functions g_k. The approximations to the solution of the boundary value problem are then taken to be the linear combinations $\sum_{k=1}^{n}\lambda_k g_k$, where the coefficients $\lambda_1, \cdots, \lambda_n$ minimize

$$\int_S [f-\sum_{k=1}^{n}\lambda_k g_k]^2 dS.$$

In the particular example of the Dirichlet problem: ($U_{xx}+U_{yy}=0$ in D; $U=f$ on S) one has the representation $U(x, y)=2\mathrm{Re}[\varphi(z)]$, where φ is analytic in $z=x+iy$ and $\mathrm{Im}[\varphi(z_0)]=0$ for some fixed z_0 in D, and the g_k are harmonic polynomials. *J. B. Diaz.*

Bertolini, Fernando. Su una generalizzazione del problema di Poisson. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 759–766.

Bei den Bemühungen, verschiedene Arten von klassischen Problemen (z.B. Rand- und Anfangswertaufgaben) zu verallgemeinern, bietet die Verwendung des Stieltjes-Integrals besondere Möglichkeiten. Der Verf. zeigt dies in der vorliegenden Arbeit am Beispiel der Poissonschen Gleichung im dreidimensionalen Falle. Die zugrundeliegende Methode, die im wesentlichen auf G. C. Evans zurückgeht, kann, wie in einer späteren Arbeit gezeigt werden soll, auf partielle Differentialgleichungen vom elliptischen Typ ausgedehnt werden. Der erste der beiden betrachteten Sätze befaßt sich mit der verallgemeinerten Poissonschen Gleichung $\int_S \partial u/\partial n_P\, d\sigma=\bar{a}(S)$ (S eine meßbare Fläche, $\bar{a}(S)$ eine additive Intervallfunktion von beschränkter Schwankung) und zeigt, daß sich auch hier die Lösung $u(P)$ in der Gestalt $u_0(P)+u_1(P)$ darstellen läßt, wobei $u_0(P)$ ein Raumpotential und $u_1(P)$ fast überall harmonisch ist. Der zweite Satz gibt eine Reihe von Formeln, die für die betrachteten Funktionen Verallgemeinerungen der klassischen Greenschen Formeln darstellen. Im zweidimensionalen Falle wurden die entsprechenden Sätze von Evans gefunden; die Beweise gehen hier zum Teil neuartige Wege. *K. Maruhn.*

See also: Craig, p. 798; Aržanyh, p. 808; Rothe, p. 808; Weinberger, p. 826; Halilov, p. 827; van Wijngaarden, p. 827; Douglas, p. 827; Basilevich, p. 838; Pisarenko, p. 840; Oswatitsch, p. 844; Vincenti, p. 845; Albring, p. 846; Freeman, p. 858; Blinova, p. 858.

Difference Equations, Functional Equations

Mitrinovitch, Dragoslav S. Sur un procédé fournissant des équations fonctionnelles dont les solutions continues et différentiables peuvent être déterminées. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. **5** (1956), 8 pp. (Serbo-Croatian summary)

Continuing earlier work [C. R. Acad. Sci. Paris **237** (1953), 550–551; MR **15**, 317] a process is developed for obtaining all solutions having suitable order of differentiability for numerous functional equations. Examples:

$$f(x)f(y)=f(x)+f^{(n)}(y) \quad (f^{(n)}\equiv n\text{th derivative}),$$

$$f^{(m)}(x)f^{(n)}(y)=af^{(p)}(x)+bf^{(q)}(y)+c \quad (a, b, c \text{ constants}),$$

$$[f(x)+Ag(x)][f(y)+Bg(y)]=f(x)+f(y)$$

(A, B constants and f, g the unknown functions).

I. M. Sheffer.

Acel', Ya. Some general methods in the theory of functional equations in one variable. New applications of fuctional equations. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 3–68. (Russian)

The author discusses several classes of functional equations, deriving properties, giving examples, and making applications to the treatment of such diverse notions as noneuclidean distance, Poisson distributions, and scalar and vector multiplication of vectors. Thus it is shown that if the functional equation

$$f[F(x, y)]=G[f(x), f(y)],$$

in which F and G are continuous and strictly monotonic, has a continuous and strictly monotonic solution f, and if one of F and G is bisymmetric, then the other also must be bisymmetric; conditions for uniqueness of the solution also are indicated. Examples given are

$$f[xy-(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}]=f(x)+f(y),$$

with solution $f(x)=c \arccos x$, and

$$f[xy-(x^2-1)^{\frac{1}{2}}(y^2-1)^{\frac{1}{2}}]=f(x)+f(y),$$

with solution $f(x)=c\ \mathrm{arc\,ch}\, x$. These results lead to characterizations of distance functions,

$$d(X, Y)=c \arccos\left[\frac{x_1y_1+x_2y_2+x_3y_3}{(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}(y_1^2+y_2^2+y_3^2)^{\frac{1}{2}}}\right],$$

$$d(X, Y)=c\ \mathrm{arc\,ch}\left[\frac{x_1y_1-x_2y_2-x_3y_3}{(x_1^2-x_2^2-x_3^2)^{\frac{1}{2}}(y_1^2-y_2^2-y_3^2)^{\frac{1}{2}}}\right],$$

in elliptic geometry and in hyperbolic geometry, respectively. *E. F. Beckenbach* (Los Angeles, Calif.).

McKiernan, Michel A. The functional differential equation $Df=1/ff$. Proc. Amer. Math. Soc. **8** (1957), 230–233.

For f, g any two functions, fg means the substitution of g into f; $f\cdot g$ is the product of f and g. Then

$$D(fg)=(Df)g\cdot Dg.$$

Let f^{-1} be the inverse function of f, j the identity function. [The notation is that of K. Menger, Calculus$\cdots$, Ginn, New York, 1955; MR **17**, 351.] Then $ff^{-1}=j$, and

$$(1) \qquad D(f^{-1}f)=1=(Df^{-1})f\cdot Df,$$

hence

$$Df=1/(Df^{-1})f.$$

The problem studied is that of analytic solutions of

(2) $$Df=1/ff, \quad Df^{-1}=f.$$

The second follows from the first by means of (1). Consider solutions which leave the complex number k fixed, i.e. $f_k(k)=k$. Then by (2), $(Df_k)k=1/k$. Let

(3) $$f_k(x)=\sum_{n=0}^{\infty} P_n(k)\cdot(x-k)^n,\ P_0(k)=k,\ P_1(k)=1/k,$$
$$P_n(k)=\frac{-1}{k}\sum_{r=2}^{n}\frac{1}{r}P_{r-1}(k).\ \sum P_{p_1}(k)\cdots P_{p_r}(k),$$

$p_1+\cdots+p_r=n$, $p_i>0$. It is shown that for k real, $k>0$, $P_S(k)=(-1)^{S-1}|P_S(k)|$, $S\geqq 1$. It is then shown that if (3) converges for real $k>0$, and $|x-k|<R_k$, then it converges for all real $k'\geqq k$, where $R_{k'}\geqq R_k$. It is then shown that there is $a_{n,r}>0$ such that $(-1)^{n-1}P_n(k)=k^S\sum_r a_{n,r}k^r$ $(n\geqq 1)$. This is used to show the convergence of (3) to functions $f_k(x)$ about $x=k$ which satisfy (2).

R. L. Jeffery (Kingston, Ont.).

See also: Praporgescu, p. 808.

Integral and Integrodifferential Equations

Lehner, Joseph. An unsymmetric operator arising in the theory of neutron diffusion. Comm. Pure Appl. Math. 9 (1956), 487–497.

The equation governing the transport of neutrons of one velocity and scattered isotropically is $\partial u/\partial t=-\xi\,\partial u/\partial x+\frac{1}{2}c\int u\,d\xi'$. In this paper the initial value problem is solved with the aid of a modification, due to Phillips, of the Hille-Yosida theorem. The possibility of expanding the solution in a series of eigenfunctions is investigated. In a previous publication [same Comm. 8 (1955), 217–234; MR 16, 1120] the author and G. M. Wing have shown that in a slab geometry there are only a finite number of characteristic modes, the corresponding eigenvalues are real and simple, and their number increases indefinitely with c. In this paper the spherical geometry is studied; since the sphere is compact, the dependence of the neutron distribution after a sufficiently long time on the initial distribution is completely continuous and the spectrum (unlike for the slab) is a pure point spectrum. The question whether a convergent series expansion can be obtained for t large enough is left undecided.

P. D. Lax.

Aržanyh, I. S. Integral equations of the fundamental problems of the theory of a field. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 13 (1954), 19–41. (Russian)

The problem mentioned in the title is that of determining a vector valued function $v=v(x, y, z)$ satisfying the partial differential equations rot $v=\Omega$, div $v=\Theta$ in an open set Q, where Q is either the interior or the exterior of a simple smooth closed surface, and Ω and Θ are given functions; the vector v being further required to satisfy certain boundary conditions on the boundary surface S. In the author's terminology, the first boundary value problem corresponds to the conditions: $(v, e_1)=v_1$, $(v, e_2)=v_2$, on S, where v_1 and v_2 are given functions and e_1 and e_2 are unit tangent vectors to the coordinate lines of S; while the second boundary value problem corresponds to the prescription: $(v, n)=w$, on S, where n is the unit outer normal to S, and w is a given function. Explicit formulas for the vector v are given in each case, and also three types of integral equations satisfied by v. These results are based on two lemmas on the representation of an arbitrary vector on $Q+S$, the first one involving the Green's function for the Neumann problem for Q and the Fredholm resolvent for the solution of the Dirichlet problem for Q as a double layer potential, while the second one involves the fundamental solution of the equation $\Delta^2\varphi+\xi^2\varphi=0$, where Δ^2 is the Laplacian, and ξ^2 is an arbitrary constant.

J. B. Diaz.

Praporgescu, N. Sur une classe d'équations intégrales. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 8 (1956), 549–616. (Romanian. Russian and French summaries)

The author studies the class of integral equations

(1) $$p(s)\varphi(s)-\int_a^b p(s, t)\varphi(s+t)dt=q(s),$$

where the variables s and t are complex, the functions $p(s)$, $p(s, t)$, $q(s)$ are (given) analytic functions, and $\varphi(s)$ is the unknown function. Fredholm's classical results are not applicable to the integral equation (1). The procedure employed consists in studying first the difference equation

(1′) $$p(s)\,\varphi(s)-\frac{1}{m}\sum_{l=1}^{m}p(s, l\omega)\varphi(s+l\omega)=q(s),$$

following N. E. Nörlund, Differenzenrechnung [Springer, Berlin, 1924], the idea being that (1′) bears the same relation to (1) as a system of linear algebraic equations bears to Fredholm's integral equation of the second kind (upon "passing to the limit"). The first three chapters are concerned with the homogeneous form of (1) and (1′), when $q(s)\equiv 0$. Chapter 1 is concerned with the homogeneous difference equation (1′). Chapter 2 deals with a special class of equations involving certain elliptic functions, which occur naturally in the considerations of chapter 3, where (1) is solved in the case of polynomial coefficients: $p(s)=\sum_{j=0}^{p}c_js^j$, $p(s, t)=\sum_{j=0}^{p}C_j(t)(s+t)^j$, with $c_p\neq 0$, $C_p(t)\not\equiv 0$. Chapter 4 is concerned with systems of equations of type (1′) and the corresponding integral equations. Chapter 5 contains applications to equations occurring in the theory of probability, while chapter 6 is devoted to the study of the non-homogeneous equation (1).

J. B. Diaz (College Park, Md.).

Calculus of Variations

Rothe, E. H. Remarks on the application of gradient mappings to the calculus of variations and the connected boundary value problems in partial differential equations. Comm. Pure Appl. Math. 9 (1956), 551–568.

Let y be real-valued on the bounded open set D in Euclidean n-space with $y=0$ on the boundary $\dot{D}$. With $y_i=\partial y/\partial t_i$ let (1) $j(y)=\int_D f(t, y, y_i)dt$. Here f is a sum of a positive definite quadratic form, a linear form and a constant term in $\{y_i\}$ with coefficients functions of t. The minimum problem is reduced to a consideration of (2) $g(x)=x+G(x)$, where $G(x)$ is a completely continuous gradient and $g(x)$ is a gradient of a functional obtained from $j(y)$ by replacing y by a linear transform x. The complete continuity of $G(x)$ is essential and for this reason

the topology of the original Hilbert space for y must in general be modified. Here the Friedrich's device of renorming is used for this purpose and is naturally expressed in terms of x. *D. G. Bourgin* (Urbana, Ill.).

Martin, Allan D. A singular functional. Proc. Amer. Math. Soc. **7** (1956), 1031–1035.

The author considers the functional

$$J(y)\Big|_c^b=\int_c^b (ry'^2+2qyy'+py^2)dx \quad (0<c<b),$$

where $r(x)$, $p(x)$, $q(x)$ are continuous in $(0, +\infty)$ and r is positive, and the integral is an L-integral. The present paper extends a previous one of W. Leighton and A. D. Martin [Trans. Amer. Math. Soc. **78** (1955), 98–128; MR **16**, 598] concerning the functional J with $q=0$. A function $y(x)$ is said to belong to the class $F[0, b]$ if $y(x)$ is AC and y'^2 is L-integrable on each closed subinterval of $(0, b]$; to the class $F'[0, b]$ if $y \in F$ and $y(x)$ is bounded in $[0, b]$; to the class $F_0[0, b]$ if $y \in F$ and $x=0$ is a limit of zeros of y. Other classes are considered. The author seeks conditions in order that $\liminf J(y)|_x^b \geqq 0$ as $x \to +0$ for all $y \in F$, $y \in F'$, $y \in F_0$. Then $J(y)$ is said to possess an F-, F'-, F_0-minimum limit respectively. Here is one result. The functional J has an F_0-minimum limit on $[0, b]$ if and only if the origin is not a focal point of the y-axis, and no conjugate point of $x=0$ falls in $[0, b)$. *L. Cesari.*

Laugwitz, Detlef. Geometrische Behandlung eines inversen Problems der Variationsrechnung. Ann. Univ. Sarav. **5** (1956), 235–244 (1957).

The extremals of a Finsler space with integrand $F(x, \dot x)$, $F(x, k\dot x)=|k|F(x, \dot x)$ have equations

$$\ddot x^i+2G^i(x, \dot x)=0,$$

where

$$G^i(x, \dot x)=\tfrac12\{{}_k{}^i{}_j\}\dot x^k\dot x^j,\quad g_{ik}=\tfrac12\,\frac{\partial^2 F(x, \dot x)}{\partial \dot x^i\partial \dot x^k}.$$

The paper treats the problem: when is a given system of curves

$$(*)\qquad \ddot x^i+2\Gamma^i(x, \dot x)=0,$$

where the $\Gamma^i(x, \dot x)$ are homogeneous of order 2 in the $\dot x^i$, the set of extremals for $F(x, \dot x)$. With $\Gamma_k{}^i=\partial\Gamma^i/\partial \dot x_k$ the equations $d\eta^i=-\Gamma_k{}^i(x, \eta)dx^k$ define a parallel displacement for which the curves (*) are autoparallel. If these are the extremals for F, then the parallel displacement becomes that of Barthel [Arch. Math. **4** (1953), 346–354, 355–365; MR **15**, 556] and leaves the length of η invariant:

$$F^2(x+dx, \eta^i-\Gamma_k{}^i(x, \eta)dx^k)-F^2(x, \eta)=0.$$

Conversely, if this relation holds then (*) are the extremals for F. The parallel displacement defines a holonomy group $H(p)$ at each point p, and (*) are the extremals for F, if and only if $H(p)$ leaves F invariant. F is unique (up to a constant factor) when $H(p)$ is transitive or on the line elements at p. *H. Busemann.*

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Mostert, Paul S.; and Shields, Allen L. On the structure of semigroups on a compact manifold with boundary. Ann. of Math. (2) **65** (1957), 117–143.

Let J be a topological semigroup with identity which is homeomorphic to a segment. Then the identity must be one endpoint and the other endpoint is a zero element. The structures of all such semigroups J are determined. There are three basic types, the ordinary multiplication on $[0, 1]$, the interval $[\frac12, 1]$ endowed with the operation $x\circ y=\max\{\frac12, xy\}$, and the interval endowed with the operation $x\circ y=\min\{x, y\}$. All other possibilities are mixtures of these three basic types.

Let S be a topological semigroup with unit element 1, which, as a topological space, is a compact manifold with boundary B. It is known that then B is a compact Lie group which contains 1. In this paper a complete structure theory is obtained for such topological semigroups. It is shown that there is a subsemigroup J in the center of S containing 1, which is homeomorphic to the unit interval and that the mapping $B\times J\to S$ given by $\langle b, j\rangle\to bj$ is onto S and one-to-one except on $B\times\{0\}$, which is mapped homomorphically onto the minimal subgroup of S with kernel which is either a two element group, a circle group, or a 3-sphere group. A number of corollaries are drawn; for example, S can be assigned a differentiable structure compatible with the semigroup operation if and only if J is the usual multiplication semigroup on $[0, 1]$.

The paper concludes with a list of problems on semigroups. *A. M. Gleason* (Cambridge, Mass.).

Lie Groups and Algebras

Harish-Chandra. Differential operators on a semisimple Lie algebra. Amer. J. Math. **79** (1957), 87–120.

The chief purpose of this paper is to prove some results needed for the author's work on Fourier transforms on semisimple Lie algebras. In part these results have been announced earlier [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 252–253, 538–540; MR **18**, 218]. Theorems 1, 2, and 3 of the present paper correspond respectively to Theorem 1, its corollary and Theorem 2 of the announcement but give these results in a somewhat improved form. We shall use the terminology and notation of the cited review. In addition this paper contains a number of other results too numerous and complicated to describe in detail here. Theorem 4 is about the restriction map from functions on g_0 to function on h_0. It asserts that this map is a homeomorphism of $J(g_0)$ on $J(h_0)$, where $J(g_0)$ is the set of all members of $\mathscr{C}(g_0)$ invariant under the adjoint group and $J(h_0)$ is the set of all members of $\mathscr{C}(h_0)$ invariant under the Weyl group. Theorems 2, 3, and 4 refer to the case in which the adjoint group is compact. In section 6 the discussion used in proving these theorems is applied to get some preliminary results in the general case. Section 7 is preliminary to section 8 which contains a detailed study of a special case which will play an important role in the contemplated applications. The chief result here (Theorem 5) asserts that a certain class of distributions which one can define on h_0 consists in fact of analytic functions. *G. W. Mackey* (Cambridge, Mass.).

See also: Cartier, p. 789; Nomizu, p. 821.

Topological Vector Spaces

Haplanov, M. G. Infinite matrices in an analytic space. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 37–44. (Russian)

Let A_R and $\bar{A}_R$ denote the spaces of points $a=(a_0, a_1, \cdots)$ whose coordinates satisfy the conditions

$$\limsup |a_n|^{1/n} \leq 1/R, \quad \limsup |a_n|^{1/n} < 1/R,$$

respectively. The author indicates various ways in which such spaces can be interpreted as spaces of analytic functions. He also considers the continuous linear operators T which (collectively) map an infinite-dimensional vector space E into an infinite-dimensional vector space E_1. In order that T map A_R with $R=1$ into itself, it is necessary and sufficient that (i) the columns of its matrix $M=(a_{jn})$ have a common majorant belonging to A_1 and (ii) there exist two constants q $(0<q<1)$ and m such that $|a_{jn}|<q^n$ for $n>mj$. The author classifies the matrices M with regard to linear independence of rows and of columns, and interprets the classification in terms of the mappings T. For example, if the rows of M are linearly independent, while some of the columns are dependent, then either M has infinitely many left inverses, and T maps E on all of E_1; or M has no inverses, and T maps E on a dense subset of E_1; in both cases, infinitely many points of E are carried into one point.

The theory is applied to the problem of determining whether a system $\{f_n(z)\}=\{\sum_j a_{jn}z^j\}$ of functions analytic in $|z|<R$ constitutes a basis in $|z|<R$. The author defines a basis to be normal if, whenever some function has the expansion $\sum c_n f_n$ and $\{c_n'\}$ is a sequence such that $|c_n'|\leq|c_n|$, then some function has the expansion $\sum c_n' f_n$. A necessary and sufficient condition for normality of a basis is the existence of a unique inverse of the matrix of the system. It is not known whether every basis is normal.

In a second application, the author considers the problem of finding analytic solutions $y=\sum c_k x^k/k!$ of a differential equation of infinite order,

$$a_0(x)y=a_1(x)y'+\cdots=f(x),$$

where $a_n(x)=\sum a_{jn}x^j$ and $f(x)=\sum b_k x^k$. Let N be the matrix (a_{jn}), and let the matrix $M=(a_{jn}')$ be formed from N by the transformation

$$a_{jn}'=a_{jn}+a_{j-1,n-1}+a_{j-2,j-2}/2!+\cdots$$

$(a_{mn}=0$ if $m<0$ or $n<0)$. Then y is given by the algebraic system of equations $Mc=b$. By virtue of the isomorphism between the differential equations and their solutions on the one hand and the algebraic systems and their solutions on the other, every classification of matrices induces a classification of differential equations. The author gives detailed answers to many questions concerning the existence, uniqueness, and function-theoretic character of solutions, especially for the cases where the coefficients $a_k(x)$ are polynomials of degree at most 2 while the right-hand side $f(x)$ is an entire function of exponential type. *G. Piranian* (Ann Arbor, Mich.).

Nef, Walter. Invariante Linearformen. Math. Nachr. **15** (1956), 123–140.

Let $\mathfrak{M}$ be a linear space, Γ a group of linear transformations σ on $\mathfrak{M}$. The paper concerns the extension of linear forms from a subspace $\mathfrak{L}$ of $\mathfrak{M}$ to linear forms invariant under Γ on the whole space, and is a chapter in the axiomatic theory of integration of Hadwiger [Comment. Math. Helv. **28** (1954), 119–148; Arch. Math. **5** (1954), 115–122; MR **16**, 22, 345]. If $x\approx y$ means that there exist x_i, y_i in $\mathfrak{M}$ and σ_i in Γ such that $x_i=\sigma_i y_i$ and $x=\sum_{i=1}^n x_i$, $y=\sum_{i=1}^n y_i$, then a linear form on a subspace $\mathfrak{L}$ can be extended to an invariant linear form on $\mathfrak{M}$ if and only if it is invariant under $\approx$. If $\mathfrak{M}$ is a partially ordered linear space satisfying the postulate that $x\geqq y$ implies $\sigma x\geqq\sigma y$ for all σ of Γ, then the order relation $\geqq$ generates a secondary partial order: $x\gtrsim y$ if there exist x' and y' such that $x\approx x'\geqq y'\approx y$. This produces an equivalence relation: $x\sim y$ if both $x\gtrsim y$ and $y\gtrsim x$. The question of extension of a linear form to an invariant monotone (positive) form is then reduced to that of monotone forms on the quotient space $(\mathfrak{M}+\mathfrak{H})/\mathfrak{H}$, where $\mathfrak{H}$ is the class of elements $x\sim 0$ [Nef, Monatsh. Math. **60** (1956), 190–197; MR **18**, 321]. In particular a linear form f_0 on a subspace $\mathfrak{L}$ of $\mathfrak{M}$ can be so extended if and only if it is monotone on $\mathfrak{L}$ in the $\sim$ sense. In the same way if $\mathfrak{M}$ is topological, or topological and partially ordered, extension properties involving invariance are deduced from corresponding results for linear spaces as given for instance in Bourbaki, 'Espaces vectorielles topologiques'. *T. H. Hildebrandt.*

Matthes, Klaus. Über eine Verallgemeinerung eines Satzes von Gelfand und Kolmogoroff. Math. Nachr. **15** (1956), 117–121.

Let $\mathfrak{M}_n$ denote a differentiable manifold of class C^n $(1\leqq n\leqq\infty)$. The main result of the paper is that the ring of all n-times continuously differentiable real-valued functions on $\mathfrak{M}_n$ that are each constant outside some compact set determines $\mathfrak{M}_n$ to within an n-times continuously differentiable homeomorphism. The author seems unaware of the overlap between his work and that of S. Myers [Proc. Amer. Math. Soc. **5** (1954), 917–922; MR **16**, 491], and L. Pursell [Pacific J. Math. **5** (1955), 963–969; MR **18**, 714]. *M. Henriksen* (Princeton, N.J.).

Robertson, Alex P.; and Robertson, Wendy. On the closed graph theorem. Proc. Glasgow Math. Assoc. **3** (1956), 9–12.

It is proved that a linear mapping of E into F with a graph closed in $E\times F$ is continuous if either (α) E is barrelled and F is fully complete, or (β) E is an inductive limit of convex Baire spaces and F is a separated inductive limit of a sequence of fully complete spaces. If E and F are separated, it is shown that a continuous linear map of F onto E is open provided that either (α) or (β) holds. *H. D. Block* (Ithaca, N.Y.).

Bonsall, F. F. Regular ideals of partially ordered vector spaces. Proc. London Math. Soc. (3) **6** (1956), 626–640.

An order ideal in a partially ordered vector space V is a linear subspace J such that $j\in J$ and $-j\leqq x\leqq j$ imply $x\in J$. A linear subset V_0 of V covers V if for each x in V there is a y in V_0 such that $-y\leqq x\leqq y$. An order unit e of V is an element of V such that the line through 0 and e covers V. An order ideal J in V is regular if the partially ordered vector space V/J has an order unit.

Theorem: If V_0 covers V and J_0 is a proper regular order ideal of V_0, and if e belongs to an order unit of V_0/J_0, then J_0 is contained in a proper regular order ideal J of V and e belongs to an order unit of V/J. Corollary: If V_0 covers V, then each monotone linear functional defined on V_0 has a monotone linear extension defined on all of V. The corollary is an extension, less general than one due to the reviewer [see R. J. Silverman, Trans. Amer.

Math. Soc. **81** (1956), 411–424, p. 412; MR **18**, 492], of the monotone-extension theorem of Kreĭn [Kreĭn and Rutman, Uspehi Mat. Nauk (N.S.) **3** (1948), no. 1(23), 3–95, Th. 1.1; MR **10**, 256; **12**, 341] and of the Kantorovich-Haviland monotone-extension theorem [Shohat and Tamarkin, The problem of moments, Amer. Math. Soc. Math. Surveys, v. 1, New York, 1943; MR **5**, 5; **13**, 1138]. This and other items about regular order ideals are used to discuss Haar measure and the Hamburger moment problem; the meat of the paper is a new proof that every B^* algebra with or without unit is a C^* algebra [see the review of Fukamiya, Kumamoto J. Sci. Ser. A. **1** (1952), no. 1, 17–22; MR **14**, 884]. The basic lemma relates regular left ideals of the algebra with regular order ideals in its set of Hermitian elements. *M. M. Day.*

Cristescu, Romulus. Classes d'espaces linéaires semi-ordonnés pseudo-normés. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 291–305. (Romanian. Russian and French summaries)

Dans cet article l'auteur étudie les propriétés de certaines catégories d'espaces vectoriels ordonnés E, réticulés et munis d'un ensemble $\mathcal{P}$ de semi-normes, tel que: 1) si $x \in E$ et $p(x)=0$ pour toute $p \in \mathcal{P}$, alors $x=0$; 2) pour $p_1 \in \mathcal{P}$, $p_2 \in \mathcal{P}$ il existe $p_3 \in \mathcal{P}$ telle que $p_3 \geqq p_1$, $p_3 \geqq p_2$; 3) pour $p \in \mathcal{P}$ et $x \in E$, $y \in E$, $p(x) \leqq p(y)$ si $|x| \leqq |y|$. *C. T. Ionescu Tulcea* (Bucharest).

Pavel, Monica. Sur les produits topologiques. Com. Acad. R. P. Romîne **6** (1956), 1073–1077. (Romanian. Russian and French summaries)

In this note the author gives three theorems concerning the projection operators of a topological vector space (t.v.s.) and the extension of linear operators. The theorems are trivial, and evidently cannot be considered as being new. As an example, here is the first of them: If the subspace A_i $(i \in I)$ of the t.v.s. E_i $(i \in I)$ has a topological supplementary, then $\prod_{i \in I} A_i$, as subspace of $\prod_{i \in I} E_i$, has a topological supplementary. *C. T. Ionescu Tulcea.*

See also: Eremin, p. 798; Pol'skiĭ, p. 802; Fantappiè, p. 806; Davis, p. 812; Pugačev, p. 825; Copson, p. 848.

Banach Spaces, Banach Algebras

Salehov, D. V. On the norm of a linear functional in an Orlicz space and on a certain internal characteristic of an L_p space. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 948–950. (Russian)

If M and N are conjugate functions relative to which Orlicz spaces are defined, then for $u \in L_M^*$ and $v \in L_N^*$, the linear functional $l(u)=(u, v)=\int_G uvdx$ has a norm $\|l\|=(k(v))^{-1}\|v\|_N$. It is known that $1 \leqq k(v) \leqq 2$. If $L_M^*=L_p$, then $k(v)=p^{1/p}q^{1/q}$. Results: 1) There is a pair M, N such that $k(v)$ assumes for appropriate v any value in the closed interval [1, 2]. 2) If M and N satisfy the classical Δ_2 condition, $M(2u) \leqq kM(u)$ for some k and all u, then $k(v) \geqq 1+\alpha > 1$ and $l(u) \leqq (1+\alpha)^{-1}\|u\|_M\|v\|_N$. 3) If $k(v)=k$ $=$const, then $M(u)=cu^r$, $v>1$. This is equivalent to each of

$$\text{(a)} \qquad \int_G M(u(x))dx=1 \Rightarrow \|u\|_M=k$$

$$\text{(b)} \qquad \int_G N(v(x))dx=1 \Rightarrow \|v\|_N=k,$$

$$\text{(c)} \qquad \{u | \int_G M(u(x))dx \leqq 1\}=\{u | \; \|u\|_M \leqq k\}.$$

In the above, the results are stated for the case of meas $(G)=\infty$. Small modifications are necessary if meas $(G)<\infty$. *B. Gelbaum* (Minneapolis, Minn.).

Krasnosel'skiĭ, M. A.; and Rutickiĭ, Ya. B. Linear integral operators operating in Orlicz spaces. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. **2** (1956), 55–76. (Russian)

Following a discussion of N'-functions and special properties of these (Δ_2-conditions, etc.) there is a lengthy investigation of the conditions on $K(x, y)$ and the pair of N'-functions Φ, Ψ which insure that the operator $\int K(x, y)u(y)\,dy$ carries L_Φ^* into L_Ψ^* and is continuous, completely continuous, etc. Details are extremely refined and cannot be reproduced here. *B. Gelbaum* (Minneapolis, Minn.).

Sobolev, V. I. Orlicz spaces over sets of infinite measure. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. **2** (1956), 77–84. (Russian)

In a series of key results, the author shows that Orlicz spaces defined relative to sets of infinite measure have most of the properties which Orlicz spaces over sets of finite measure enjoy. Most of the computations are straightforward. *B. Gelbaum* (Minneapolis, Minn.).

Sakai, Shôichirô. A characterization of W^*-algebras. Pacific J. Math. **6** (1956), 763–773.

Dixmier proved [Bull. Soc. Math. France **81** (1953), 9–39; MR **15**, 539] that every W^* algebra is the dual space of a certain Banach space. The present author proves conversely that if a C^* algebra M is a dual space, then M has a faithful W^* representation. The Hilbert space on which he makes M act is the direct sum of all spaces H_φ defined by weak-star continuous states φ of norm 1. Repeated use is made of the weak-star topology on M, and in fact the author finally shows that this topology coincides, at least on bounded sets, with the weak operator topology. Thus the present characterization resembles the earliest non-spatial description of W^*-like algebras, that of von Neumann [Mat. Sb. N.S. **1(43)** (1936), 415–484]. [For a survey of other W^* characterizations see Kaplansky's review of Kadison, Ann. of Math. (2) **64** (1956), 175–181; MR **18**, 54. To Kaplansky's list should be added the forthcoming paper by J. Feldman on AW^* embeddings, abstracted in Bull. Amer. Math. Soc. **62** (1956) p. 158.] *H. Mirkil.*

Schaefer, Helmut. Über die Methode sukzessiver Approximationen. Jber. Deutsch. Math. Verein. **59** (1957), Abt. 1, 131–140.

Let $\mathfrak{T}$ be a bounded closed convex subset of a Banach space $\mathfrak{E}$ and let A be a (not necessarily linear) map of $\mathfrak{T}$ into $\mathfrak{T}$ satisfying $\|Ax-Ay\| \leqq \|x-y\|$. Let α satisfy $0<\alpha<1$ and given x_0 in $\mathfrak{T}$, set

$$x_{n+1}=\alpha Ax_n+(1-\alpha)x_n \quad (n=0, 1, 2, \cdots).$$

(I) If $\mathfrak{E}$ is strictly convex, the set of fixed points of A is convex. (II) If $\mathfrak{E}$ is uniformly convex and $A(\mathfrak{T})$ is compact, then the sequence (x_n) converges strongly to a fixed point. (III) If $\mathfrak{E}$ is real Hilbert space and A is weakly continuous on $\mathfrak{T}$, then (x_n) converges weakly to a fixed point. In general, these fixed points depend on both x_0 and α, but (IV) if A is linear, the limit is independent of α, and in case A is normal the limit is the orthogonal projection of x_0 onto the eigenspace corresponding to $\lambda=1$. When $\alpha=\frac{1}{2}$, statement (II) was proved by M. A.

Krasnosel'skiĭ [Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1(63), 123–127; MR **16**, 833]. The author relies on the fact that a Banach space is uniformly convex if and only if for some α with $0<\alpha<1$, we have (C_α): given $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon,\alpha)$ such that if $\|x\|, \|y\|\leq 1$ and $\|x-y\|>\varepsilon$ then $\|\alpha x+(1-\alpha)y\|\leq(1-\delta)\max\{\|x\|,\|y\|\}$.
R. G. Bartle (Urbana, Ill.).

Hausner, Alvin. Ideals in a certain Banach algebra. Proc. Amer. Math. Soc. **8** (1957), 246–249.

Let X be a complex commutative Banach algebra with $\mathfrak{M}$ as its space of regular maximal ideals. Let Ω be a compact Hausdorff space and $C(\Omega, X)$ be the Banach algebra of all complex continuous functions on Ω with values in X. Denote by $\mathfrak{M}_1$ the space of regular maximal ideals of $C(\Omega, X)$. It is shown that $\mathfrak{M}_1$ is homeomorphic to the product space $\Omega\times\mathfrak{M}$. *B. Yood* (Eugene, Ore.).

Coddington, Earl A. Some Banach algebras. Proc. Amer. Math. Soc. **8** (1957), 258–261.

Let S be a measure space, L^1, L^2, L^∞ the usual Lebesgue classes over S. Suppose $\{\phi_k\}\subseteq L^1\cap L^2\cap L^\infty$ is a complete orthonormal sequence in L^2 and $\{\nu_k\}$ any sequence of nonzero complex numbers such that

$$\sum_{k=1}^{\infty}\|\phi_k\|_\infty{}^2\|\phi_k\|\,|\nu_k|\leq 1.$$

For $f, g\in L^1$ define

$$f*g=\sum_{k=1}^{\infty}(f,\phi_k)(g,\phi_k)\bar{\nu}_k\phi_k,$$

where $(f,\phi_k)=\int f\bar{\phi}_k$. L^1 with $*$ as a multiplication is a regular commutative Banach algebra whose maximal ideal space may be identified with the positive integers. Sufficient conditions are given that a generalized Wiener Tauberian theorem should hold. An example is given which shows that there exists an algebra of this type which is not self-adjoint. Sufficient conditions are given that certain translates are dense in the algebra. {In several places the author makes the assumption that the set $\{\phi_k\}$ is dense in L^1. He really needs that the linear manifold determined by these elements is dense in L^1.}
A. Devinatz (St. Louis, Mo.).

Nikol'skiĭ, S. M. On a family of function spaces. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 6(72), 203–212. (Russian)

Descriptive survey of properties of the author's spaces $H_p^{(r)}(G)$, with particular regard to variation of the parameter r. *M. G. Arsove* (Seattle, Wash.).

See also: Pol'skiĭ, p. 802.

Hilbert Space

Davis, Chandler. A Schwarz inequality for convex operator functions. Proc. Amer. Math. Soc. **8** (1957), 42–44.

For any Hilbert space H let K be the set of self-adjoint operators with spectrum contained in interval I. If f is a function from K to the self-adjoint operators on H obtained from a real valued f_0 on I by $f(A)=\int f_0(\lambda)dE_\lambda$, where E_λ is the spectral resolution of A, then f is called an operator function associated with I. The operator function f associated with I is convex if for each $A, B\in K$ and $0\leq t\leq 1$ it follows that $f(tA+(1-t)B)\leq tf(A)+(1-t)f(B)$. Theorem: The operator function f associated with I is convex if and only if for any $A\in K$ and orthogonal projection P it follows that $Pf(PAP)P\leq Pf(A)P$. This settles in the affirmative a conjecture made by the reviewer to the author. *S. Sherman* (Philadelphia, Pa.).

Hill, C. K. The Hilbert bound of a certain doubly-infinite matrix. J. London Math. Soc. **32** (1957), 7–17.

The doubly-infinite matrix of the title, and the corresponding singly-infinite matrix, are

$$A(\lambda)=\left(\frac{1}{m+n+\lambda}\right)\text{ and }A_0^\infty(\lambda_0)=\left(\frac{1}{m+n+\lambda}\right),$$

where m, n range over all integers in the first case, and over all non-negative integers in the second; and in each case λ is a real constant such that no denominator vanishes. Extending a special result obtained by the reviewer for $A_0{}^\infty(\lambda)$ [same J. **11** (1936), 237–240] the author proves that $A(\lambda)$ is a Hilbert matrix with attained Hilbert bound $|\Lambda|$ for all λ, where $\Lambda=\pi/(\sin\pi\lambda)$; and that the same is true of $A_0{}^\infty(\lambda)$ for $\lambda<\frac{1}{2}$. It is proved also that the numbers $\pm\Lambda$ are latent roots of $A(\lambda)$ for all λ, and of $A_0{}^\infty(\lambda)$ for certain λ; and the sets of Hilbert root vectors associated with these two roots are investigated. In the case of $A_0{}^\infty(\lambda)$ each of these two sets (when it exists) is shown to have a finite base, and a suitable base (depending on λ) is given explicitly for each. *A. E. Ingham.*

Segal, I. E. The structure of a class of representations of the unitary group on a Hilbert space. Proc. Amer. Math. Soc. **8** (1957), 197–203.

Let G be the group of unitary operators on a complex Hilbert space X, G being topologized according to the strong operator topology, and let T be a given continuous unitary representation of G on a complex Hilbert space Y. Being guided by the particle interpretation in the quantum theory of fields, the author calls the representation T "physical" if for every orthogonal projection P in X the self-adjoint operator A_P in Y determined by the one-parameter group $\{U_t\}$ of unitary operators on Y, $U_t=T(e^{itP})=e^{itA_P}$, is non-negative. With this definition, and using an earlier paper of his [Trans. Amer. Math. Soc. **81** (1956), 106–134; MR **17**, 880], the author proves: "A continuous physical unitary representation of the unitary group on a complex Hilbert space is a direct sum of irreducible such representations. The latter are unitarily equivalent to the canonical representations of the group on the spaces of covariant tensors of maximal symmetry types."
F. H. Brownell (Seattle, Wash.).

Yen, Ti. Isomorphism of AW^*-algebras. Proc. Amer. Math. Soc. **8** (1957), 345–349.

In this note the following extension of a result of H. A. Dye [Ann. of Math. (2) **61** (1955), 73–89; MR **16**, 598] is proved: Let M and N be two AW^*-algebras having no component of type I_n $(n=1, 2, \cdots)$. Let ρ be an algebraic isomorphism of the unitary group M_u of M onto the unitary group N_u of N. Then ρ induces an orthoisomorphism θ (isomorphism which preserves orthogonality) between the projection lattices M_P and N_P of M and N respectively. His method of proof is similar to Dye's and consists in examining the operators $\rho(\lambda e+1-e)$, $|\lambda|=1$, e a fixed projection, whereas θ is defined by the equation $\rho(1-2e)=1-2\theta(e)$. *W. A. J. Luxemburg.*

See also: Kato, p. 786; Rothe, p. 808; Sakai, p. 811; Pugačev, p. 825; Vachaspati, p. 851.

TOPOLOGY

General Topology

Besicovitch, A. S. On families of domains. Proc. Cambridge Philos. Soc. **53** (1957), 73–75.

The author considers uniformly bounded simply connected plane domains $D(x)$ which depend on a real parameter x and increase with x (this means that the closure of $D(x)$ is a subset of $D(x^1)$ for each $x^1>x$). The main question discussed is whether the boundary of $D(x)$ is a Jordan curve for almost all x. The answer turns out to be negative unless a further hypothesis is made. However a weaker statement is established with ease.

L. C. Young (Madison, Wis.).

Pannoli Massaro, Giliana. Quelques questions préalables à propos du problème des sélections, en rapport avec celui des fonctions implicites. C. R. Acad. Sci. Paris **244** (1957), 153–155.

Let E be a continuous, multivalued function from an interval of the real line to the real line. Let (x_0, y_0) be a point in the graph of E. This point is said to be accessible if there is a neighborhood I of x_0 and a continuous single-valued function f defined in I such that $f(x_0)=y_0$ and $f(x) \in E(x)$, for all x in I. Theorem: The set of non-accessible points in the graph of E is dense-in-itself.

E. G. Begle (New Haven, Conn.).

Han Hen Gon. On certain classes of topological spaces. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 959–961. (Russian)

Let $\mathfrak{a}$ be a regular cardinal number. A system of subsets γ of a topological space P is called an $L_{\mathfrak{a}}$-system if every point of P has an open neighborhood that intersects fewer than $\mathfrak{a}$ sets of the system γ. A topological space P is said to belong to the class $T_{\mathfrak{a}}$ if it has an open basis that can be written as the union of fewer than $\mathfrak{a}$ $L_{\mathfrak{a}}$-systems. P is said to belong to the class $R_{\mathfrak{a}}$ if it has an open basis that can be written as the union of fewer than $\mathfrak{a}$ locally finite systems. (Note that $R_{\mathfrak{a}} \Rightarrow T_{\mathfrak{a}}$ trivially.) For $\mathfrak{a}=\aleph_1$, Smirnov's metrization theorem [Uspehi Mat. Nauk (N.S.) **6** (1951), no. 6(46), 100–111; MR **14**, 70] shows that the class $R_{\mathfrak{a}}$ is just the class of metrizable spaces.

Theorem: The following 6 properties are equivalent for topological spaces P of the classes $T_{\mathfrak{a}}$ (and $R_{\mathfrak{a}}$). 1. The weight of P is less than $\mathfrak{a}$. 2. Every well-ordered decreasing sequence of order type $\omega(\mathfrak{a})$ of closed sets is stationary. 3. Every well-ordered increasing sequence of order type $\omega(\mathfrak{a})$ of closed sets is stationary. 4. Every system of pairwise disjoint (non-void) open sets has cardinal number less than $\mathfrak{a}$. 5. Every open covering of P admits a subcovering of cardinal number less than $\mathfrak{a}$. 6. Every open covering of P of cardinal number $\mathfrak{a}$ admits a subcovering of cardinal number less than $\mathfrak{a}$.

For $\mathfrak{a}=\aleph_1$ and P metrizable, the theorem is well-known [Aleksandrov and Uryson, Trudy Mat. Inst. Steklov. **31** (1950), p. 76; MR **13**, 264]. The proof is sketched and several corollaries are stated.

E. Hewitt.

Smirnov, Yu. M. On the metrizability of bicompacts decomposable into a sum of sets with countable basis. Fund. Math. **43** (1956), 387–393. (Russian)

Let R be a locally countably compact Hausdorff space that can be written as the union of a countable number of subspaces each having a countable basis. Then R itself has a countable basis and is hence metrizable. This interesting, and general, result shows in particular that no bicompact non-metrizable Hausdorff space is the union of two metrizable subspaces, and hence answers in the negative a question raised by Aleksandrov and Uryson [Trudy Mat. Inst. Steklov. **31** (1950), p. 85; MR **13**, 264].

E. Hewitt (Seattle, Wash.).

Aleksandrov, P. On two theorems of Yu. Smirnov in the theory of bicompact extensions. Fund. Math. **43** (1956), 394–398. (Russian)

Let R be a completely regular T_0-space. Yu. Smirnov has proved the following two theorems [Mat. Sb. N.S. **31**(73) (1952), 152–166; MR **14**, 303]. (I) Two bicompact extensions $b'R$ and $b''R$ of R are different if and only if R contains two closed subsets whose closures in one of the extensions intersect and whose closures in the other extension do not intersect. (II) The space R admits at least two different bicompact extensions if and only if R contains two functionally separated non-bicompact closed subsets. [A special case of (II) was proved by the reviewer, Trans. Amer. Math. Soc. **64** (1948), 45–99; MR **10**, 125.] Very simple proofs of theorems (I) and (II) are given in the present paper.

E. Hewitt (Seattle, Wash.).

Mruvka, S. Remark on P. Aleksandrov's work "On two theorems of Yu. Smirnov". Fund. Math. **43** (1956), 399–400. (Russian)

Still another proof of theorem (I) of the paper reviewed above.

E. Hewitt (Seattle, Wash.).

Weier, Josef. Über einen Erweiterungssatz. Monatsh. Math. **61** (1957), 51–53.

The following theorem is proved. Let P be a compact metric space, Q a compact topological manifold, A a closed subset of P and f a continuous mapping of A into Q. Then there exists an open set U with $A \subseteq U \subseteq P$ such that f has a continuous extension over $\bar{U}$.

H. D. Block.

See also: Ivašev-Musatov, p. 803; Hilton, p. 814.

Algebraic Topology

Bokšteĭn, M. F. Homology invariants of topological spaces. Trudy Moskov. Mat. Obšč. **5** (1956), 3–80. (Russian)

This paper is concerned with the relations between the cohomology rings (∇-rings) of a locally compact Hausdorff space A over various coefficient rings. Aleksandrov's definition of cohomology (essentially cohomology with compact carriers) is used. Let Z_k denote the integers $(=Z_0)$ reduced mod k. Let m divide m' ($m>0$, $m'\geq 0$); the obvious maps of $Z_{m'}$ onto Z_m, resp. of Z_m into $Z_{m'}$, induce maps of the cohomology rings over these groups, denoted by $\pi_m{}^{m'}$, resp. $\bar{\omega}_{m'}{}^m$. The π's are multiplicative, the $\bar{\omega}$'s satisfy a somewhat different law. The set of ∇-rings $H^*(A, Z_k)$ ($k=0, 1, 2, \cdots$) together with all the π's and $\bar{\omega}$'s is called the modular cohom.-spectrum of A. Paragraph 2 discusses two three-manifolds whose integral ∇-rings are isomorphic, but whose ∇-rings over Z_2 are different. Paragraph 3 proves that the modular spectrum

determines the ∇-ring over any coefficient ring R. The ring $H^*(A, R)$ is constructed by an algebraic procedure similar to the tensor product. It begins with formal sums $\sum r_i u_i(m_i)$, where $u_i(m_i) \in H^*(A, Z_{m_i})$, $r_i \in R$, and $m_i r_i = 0$. Multiplication and certain identifications of these sums are introduced, with the π's and $\bar{\omega}$'s intervening. Paragraph 4 gives a criterion for such a formal sum to be equivalent to 0 under the identifications. Paragraph 5 brings a proof of the universality of integral cohomology for the cohomology groups ($\cup$-product omitted), essentially Steenrod's theorem. Use is made of the Bokšteĭn-Whitney operator, associated with the exact coefficient sequences $0 \to Z_0 \to Z_0 \to Z_m \to 0$ and $0 \to Z_{\tilde{m}} \to Z_{m'} \to Z_m \to 0$ and the induced exact cohom.-triangles (although not in this language). Ch. II concerns dimension (cohom.-dimension) $\mathrm{Dim}_G A$ over a group G, defined as the largest q for which there exists an open set A' with $H^q(A', G) \neq 0$. The main theorem states that $\mathrm{Dim}_G A$ can be computed for arbitrary G, if it is known for the following G's: The rationals, the p-adic rationals (denominator prime to p), the cyclic groups Z_p, and the groups Z_{p^∞} (all roots of unity of order a power of p), p running through all primes. [A set of groups which determines $\mathrm{Dim}_G A$ for arbitrary G is called a full system of coefficient groups for cohomological dimension.] A sharper statement is made: To an arbitrary G one associates a subset S_G of the groups listed above, depending on whether G contains elements of infinite order, elements of order p, etc.; $\mathrm{Dim}_G A$ is simply $\max\{\mathrm{Dim}_H A : H \in S_G\}$. The proof rests on lemmas of the following form: If G and $H^q(A, Z)$ contain elements of infinite order, then $H^q(A, G) \neq 0$. The constructions of paragraphs 3, 4 are used here. — A second part, to appear in the same Trudy will treat product spaces. *H. Samelson.*

Postnikov, M. M. Squares of classes of contrahomologies. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **22** (1956), 127–164. (Russian)

This is an expository article (containing part of the author's 1949 candidate's dissertation) on the Steenrod $\cup_i$-products and their applications. Let X be an $(n-1)$-connected space; the author describes a problem relating to the space X as a first-stage problem (there are zero-stage problems) if it depends only on $\pi_n(X)$, $\pi_{n+1}(X)$ and the Eilenberg-MacLane invariant k belonging to $H^{n+2}(K(\pi_n(X); n); \pi_{n+1}(X))$. Thus, for example, the determination of $H^{n+1}(X)$ and the homotopy classification of maps of $(n+1)$-dimensional polyhedra into X are first-stage problems. The author regards a first-stage problem as virtually solved when k is given in explicit form by means of Steenrod operations; this is achieved in the final section.

The paper contains a detailed account (sections 1, 2 and 5) of $\cup_i$-products in semisimplicial complexes. In section 4 group pairings and heteromorphisms are discussed. A group pairing of G to H is a map $\theta: G \times G \to H$ such that $\theta(g_1, g_2) = \theta(g_2, g_1)$ and $\theta(g, g_1 + g_2) = \theta(g, g_1) + \theta(g, g_2)$; it is dyadic if $2\theta = 0$. A heteromorphism is a map $\eta: G \to H$ such that $\eta(g) = \eta(-g)$ and $\eta(g_1+g_2+g_3) + \eta(g_1) + \eta(g_2) + \eta(g_3) = \eta(g_1+g_2) + \eta(g_1+g_3) + \eta(g_2+g_3)$ [cf. Eilenberg and MacLane, Ann. of Math. (2) **60** (1954), 49–139; MR **16**, 391]. To a dyadic pairing θ belongs a (dyadic) homomorphism η given by $\eta(g) = \theta(g, g)$ and to a heteromorphism η belongs a pairing θ given by $\theta(g_1, g_2) = \eta(g_1) + \eta(g_2) - \eta(g_1 + g_2)$. (All groups are abelian.) The Pontryagin heteromorphism is the map $\eta: \pi_n(X) \to \pi_{n+1}(X)$ defined by composition with a generator of $\pi_{n+1}(S^n)$, $n > 1$.

If x^p is an integer valued cochain of K, then we write $x^p \equiv 0 \pmod{a}$ if, for each $\sigma \in K$, $x^p(\sigma)$ is divisible by the integer a. Let

$$\omega_i(x^p) = x^p \cup_{p-i} x^p + x^p \cup_{p-i+1} \delta x^p.$$

Then $\omega_i(x^p) \equiv 0 \pmod{\tilde{a}}$ if $x^p \equiv 0 \pmod{a}$, where

$$\tilde{a} = \begin{cases} 2a, & \text{if } a \text{ is even and if either } i \text{ is odd, or } i \geqq p; \\ a, & \text{if } a \text{ is odd and if either } i \text{ is odd, or } i \geqq p; \\ 2, & \text{if } a \text{ is even and } i \text{ is even and } i < p; \\ 1, & \text{if } a \text{ is odd and } i \text{ is even and } i < p. \end{cases}$$

Moreover $\delta x^p \equiv 0 \pmod{a}$ if $\delta\omega_i(x^p) \equiv 0 \pmod{\tilde{a}}$, so that we obtain an operation

$$Kv^i: H^p(K, L; \bmod a) \to H^{p+i}(K, L; \bmod \tilde{a}),$$

called by the author the Pontryagin square. This may be further generalized to yield an operation

$$Kv_\eta{}^i: H^p(K, L; G) \to H^{p+i}(K, L; H)$$

associated with an arbitrary heteromorphism $\eta: G \to H$. If u is the fundamental class $\in H^n(K(\pi_n(X); n); \pi_n(X))$ and if η is the Pontryagin heteromorphism, then $k = Kv_\eta{}^i u$.

Astonishingly, the article contains no reference to Steenrod's fundamental paper [Ann. of Math. (2) **48** (1947), 290–320; MR **9**, 154]. A happier innovation is the use of 'contrahomology' for 'cohomology', which has much to recommend it. *P. J. Hilton* (Manchester).

Hilton, P. J. On divisors and multiples of continuous maps. Fund. Math. **43** (1956), 358–386.

The author generalizes Borsuk's notion of dependence [Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 81–85, 251–254; MR **16**, 946; **17**, 652] as follows: let X_0 be a compact space, and Y_0, Y_1 be compact ANR or infinite polyhedra. Let Φ be a collection of maps $X_0 \to Y_0$ and g a map $X_0 \to Y_1$ such that, for all compact spaces $X \supset X_0$, the existence of an extension of f to a map $X \to Y_0$, for every $f \in \Phi$, implies the existence of an extension of g to a map $X \to Y_1$. Then g is said to be dependent on Φ. Several variations of the above notion are also discussed; a connection with Thom's notion of the dependence of a cohomology class upon another [Séminaire Henri Cartan de l'Ecole Norm. Sup., 1954/1955, Paris, 1955, Exposé 17] is established. The above definition depends only on the homotopy classes of the maps concerned.

Let $\pi(A; B)$ denote the set of homotopy classes of maps $(A, a_0) \to (B, b_0)$, and X_0, Y_0, Y_1 be as above. Consider $\alpha \in \pi(X_0, Y_0)$; α determines a function α^*: $\pi(Y_0, Y_1) \to \pi(X_0, Y_1)$ by composition. The author studies the various cases when $\pi(X_0, Y_1)$ and $\pi(Y_0, Y_1)$ are groups and determines the relationship between $\alpha^*\pi(Y_0, Y_1)$ and the subset of $\pi(X_0, Y_1)$ consisting of those elements which are dependent on α.

Finally, the main theorems of Borsuk are generalized, and several problems left open by him are solved. *J.-P. Meyer* (Baltimore, Md.).

Chang, S. C. On algebraic structures and homotopy invariants. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 797–800 (1957).

The author discusses very briefly various results on A_n^2-polyhedra obtained by J. H. C. Whitehead [Ann. Soc. Polon. Math. **21** (1948), 176–186; MR **11**, 48], P. J. Hilton [Proc. London Math. Soc. (3) **1** (1951), 462–493; MR **13**, 674] and himself [Proc. Roy. Soc. London. Ser. A. **202** (1950), 253–263; MR **12**, 120], and recalls the generalization of secondary torsions (due to the author) ob-

tained by himself and Whitehead [Quart. J. Math. Oxford Ser. (2) 2 (1951), 167–174; MR 13, 374]. He announces further generalizations, enabling him to classify cohomology systems involving two Steenrod squares; as a special case, complete numerical invariants are found, which characterize the homotopy type of A_n^3-polyhedra ($n\geqq 3$).

No details whatsoever are given. An A_n^k-polyhedron K is an arcwise-connected polyhedron of dimension $\leqq n+k$ and such that $\pi_i(K)=0$ ($i=1, 2, \cdots, n-1$).

J.-P. Meyer (Baltimore, Md.).

Inoue, Yoshiro. A complete set of invariants of the singular homotopy type. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 28–38.

All spaces considered here are arc-connected. Call X' n-combined with X if X' has the same singular n-homotopy type [P. Olum, Ann. of Math. (2) **60** (1954), 317–325; MR **16**, 159] as X and $\pi_i(X')=0$ ($i\geqq n+1$); X' can be obtained by attaching cells of dimension $>n+1$ to X and is unique up to a singular homotopy equivalence. It is written $X_{(n)}$ and $k^{n+2}(X)\in H^{n+2}(X_{(n)}, \pi_{n+1}(X))$ is the obstruction to the retraction of $X_{(n)}$ onto X. Let $M_{(n)}(X)$ be the minimal complex of $X_{(n)}$; spaces with the same singular n-homotopy type have isomorphic $M_{(n)}(X)$. The author gives an explicit construction of $M_{(n+1)}(X)$ by using only $M_{(n)}$, $\pi_{n+1}(X)$, and $k^{n+2}(X)$. As a first application, it is proved that: X and Y have the same singular $(n+1)$-homotopy type if and only if they have the same singular n-homotopy type and there is an isomorphism $\theta:\pi_{n+1}(X)\approx\pi_{n+1}(Y)$ such that, under the induced coefficient group isomorphism

$$\theta^{\#}:H^{n+2}(M_{(n)},\pi_{n+1}(X))\approx H^{n+2}(M_{(n)}, \pi_{n+1}(Y)),$$
$$\theta^{\#}(k^{n+2}(X))=k^{n+2}(Y).$$

Induction on n leads the author to a set of algebraic invariants which completely characterize singular homotopy type. As a second application, the author shows: Let G be an abelian group. Then a suitable filtration of the chains $C(M_{(n+1)}(X), G)$ leads to a spectral sequence such that $E_2^{p,q}$ is canonically isomorphic to

$$H_p(M_{(n)}(X), H_q(\pi_{n+1}(X); n+1; G)).$$

J. Dugundji (Los Angeles, Calif.).

Peterson, Franklin P. Generalized cohomotopy groups. Amer. J. Math. **78** (1956), 259–281.

In this paper the author introduces the notion of cohomotopy groups with coefficients in an abelian group G. Let K be a complex of dimension N, and let $\pi^n(K; G)$ denote the n-dimensional cohomotopy group of K with coefficients in G. This group is defined if $N\leqq 2n-2$, and corresponds in a one-to-one fashion with the homotopy classes of maps of K into any simply connected space X, such that all its positive-dimensional homology is zero except in dimension n, and its n-dimensional homology group is G. The group $\pi^n(K; Z)$, where Z is the group of integers, is just the ordinary n-dimensional cohomotopy group of K, $\pi^n(K)$.

The properties of the groups $\pi^n(K; G)$ are similar to those of ordinary cohomology groups. Further, if G has no elements of order 2, then any homomorphism ϕ: $G\to H$ induces a homomorphism $\phi_{\#}$: $\pi^n(K; G)\to\pi^n(K; H)$. Moreover if $0\to A\to B\to C\to 0$ is an exact sequence of abelian groups having no elements of order 2, then there is an exact sequence

$$\cdots\to\pi^n(K; A)\to\pi^n(K; B)\to\pi^n(K; C)\to\pi^{n+1}(K; A)\to\cdots.$$

Finally the author proves a universal coefficient theorem which asserts that there is a split exact sequence

$$0\to\pi^n(K)\otimes G\to\pi^n(K; G)\to\mathrm{Tor}(\pi^{n+1}(K), G)\to 0.$$

J. C. Moore (Princeton, N.J.).

★ **Fuller, F. B. A relation between degree and linking numbers.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 258–262. Princeton University Press, Princeton, N. J., 1957. $7.50.

In this note, the author uses the imbedding method to derive the Lefschetz fixed-point formula from a relation involving linking numbers. This relation is stated as follows.

Let f and g be two mappings from an oriented n-manifold M into the Euclidean $(n+1)$-space E^{n+1} whose images fM and gM are disjoint. The vectors

$$[g(x)-f(x)]/|g(x)-f(x)|$$

define a mapping $|f, g|$ of M into the unit sphere S^n with center at the origin. Let the cycles u_i^p form a basis for the rational p-cycle classes on M and let S^n be naturally oriented by E^{n+1}. Then the degree of the mapping $|f, g|$ is equal to

$$\sum(-1)^{p+1}\varepsilon_{ij}{}^pLk(f_{\#}u_i{}^p, g_{\#}u_j{}^{n-p}),$$

where the matrix $(\varepsilon_{ij}{}^p)$ of coefficients is the transposed inverse of the intersection matrix $[(u_i{}^p, u_j{}^{n-p})]$ and $Lk(f_{\#}u_i{}^p, g_{\#}u_j{}^{n-p})$ is the linking number of the disjoint singular cycles $f_{\#}u_i{}^p$ and $g_{\#}u_j{}^{n-p}$. *S. T. Hu.*

Milnor, John. The geometric realization of a semi-simplicial complex. Ann. of Math. (2) **65** (1957), 357–362.

The author proves a new geometric realization theorem for semi-simplicial complexes in this paper. This geometric realization is different from those used earlier by Giever [Ann. of Math. (2) **51** (1950), 178–191; MR **11**, 379] and Hu [Pacific J. Math. **1** (1951), 583–602; MR **13**, 676] in that degenerate simplexes are collapsed. Let $|K|$ denote the geometric realization of the semi-simplicial complex K. The following theorem is proved. There is a one to one continuous map η: $|K\times L|\to|K|\times|L|$ which is onto, and which is a homeomorphism if either a) K and L are countable, or b) K or L is locally finite. In this theorem $K\times L$ denotes the Cartesian product of the semi-simplicial complexes K and L, and $|K|\times|L|$ the product of the topological spaces $|K|$ and $|L|$. No such theorem is true for the earlier geometric realizations, and this is one of the main reasons for interest in this particular geometric realization. One application is that if $K(\pi, n)$ is the explicit complex of Eilenberg and MacLane [Ann. of Math. **51** (1950), 514–533; MR **11**, 735] and π is a countable abelian group, then $|K(\pi, n)|$ is an abelian topological group.

For any topological space X, let $S(X)$ denote the singular complex of X. We may now state some further properties of the geometric realization functor. For any semi-simplical complex K, there is a natural map i: $K\to S(|K|)$ which induces a homology isomorphism, and for any topological space X, there is a natural map j:$|S(X)|\to X$ which induces an isomorphism of singular homology and homotopy groups. *J. C. Moore.*

See also: Matthes, p. 810; Fary, p. 822.

GEOMETRY

Geometries, Euclidean and other

Havel, Václav. On a theorem of Kadeřávek. Časopis Pěst. Mat. **80** (1955), 328–330. (Czech)

Havel, Václav. On wedge-shaped surfaces. II. Časopis Pěst. Mat. **80** (1955), 308–316. (Czech)

Balashov, V. The choice of the unit cell in the triclinic system. Acta Cryst. **9** (1956), 319–320.

The following rules for choice and evaluation of the unit cell in the triclinic system are set up: 1. The unit cell should be primitive (i.e. it should have the smallest volume). 2. The cell edges should be labelled in the following order: $a<b<c$. 3. The interaxial angles α, β and γ should be "homogeneous" (that is either all acute or all obtuse). 4. The Delaunay reduction is applied in direct space to any primitive cell obtained experimentally. 5. The edges a, b and c of the cell obtained after operation (4) are tested to determine whether they are shorter than the diagonals of the faces bordering the cell. 6. If this test reveals a diagonal shorter than an edge, it must be concluded that the shortest translations a, b and c form the acute parallelepiped: the Delaunay reduction should be applied in reciprocal space to the values a^*, b^*, $\cdots$ etc. 7. From the reduced parameters a^*, b^*, $\cdots$ etc. obtained after operation (6), the corresponding values a, b, $\cdots$ etc. are calculated and accepted as parameters of the direct lattice. 8. When one or two interaxial angles are equal to 90°, under the conditions of the experiment, the unit cell should be presented as the non-acute (α, β, $\gamma \geqq 90°$) paralellepiped. For the sake of completeness it may be classed as an obtuse parallelepiped. By the application of the above rules, all triclinic lattices (direct space) will therefore be divided into two groups, for one of which the unit cell is an obtuse parallelepiped and for the other of which the unit cell is an acute parallelepiped. In the former case the translations a, b, and c of the unit cell are the shortest possible. In the latter case, the translations a, b and c are not necessarily the shortest possible, but they will usually be so, and the interaxial angles will usually be not greatly different from 90°. Moreover, they have a unique definition and are automatically derived by the application of the Delaunay reduction in reciprocal space. The application of the above rules thus avoids the inconvenient values of lattice parameters obtained in many cases when the Delaunay reduction is carried out in one (direct or reciprocal) space only. *W. Nowacki.*

Hall, Marshall, Jr.; Swift, J. Dean; and Walker, Robert J. Uniqueness of the projective plane of order eight. Math. Tables Aids Comput. **10** (1956), 186–194.

A finite projective plane with $n+1$ points per line is said to be of order n. This paper demonstrates the uniqueness of the projective plane of order 8. This result combined with the previous literature implies that 9 is the smallest order of a non-Desarguesian plane. The demonstration of the uniqueness of the plane of order 8 utilizes the Norton list of Latin squares of order 7. This list includes a total of 147 distinct varieties, an omission in the original Norton list of 146 varieties having been found by Sade. These 147 varieties form the starting point for the search for planes of order 8. However, it can be shown that only 100 of these require actual consideration. The computer SWAC is used to make 100 separate runs for each of the 7 by 7 squares under consideration. Details of the run and coding appear in Section 4 of the paper, and Section 5 shows that the SWAC data is of such a nature that it is relatively easy to complete the search by hand. *H. J. Ryser* (Columbus, Ohio).

Khan, Nisar A. On incidence matrices. Ganita **5** (1954), 117–122 (1955).

The author discusses properties of the incidence matrix A of a v, k, λ configuration. The result in Section 2 may be strengthened to assert that one characteristic root of A equals k, and the remaining are each equal to $\sqrt{(k-\lambda)}$ in absolute value. [See Ryser, Amer. Math. Monthly **62** (1955), no. 7, part II, 25–31, p. 30; MR **17**, 401.] *H. J. Ryser* (Columbus, Ohio).

Sasayama, Hiroyoshi. On n-dimensional generalization of the quasi euclidean space. Comment. Math. Univ. St. Paul. **5** (1956), 95–114.

The quasi-euclidean distance from a point $P(x^i)$ to a point $Q(y^i)$ is defined by the cyclical determinant

$$\rho(P,Q)=\begin{vmatrix} y^1-x^1, & y^2-x^2, & \dots, & y^n-x^n \\ y^n-x^n, & y^1-x^1, & \dots, & y^{n-1}-x^{n-1} \\ \cdot & \cdot & \dots & \cdot \\ y^2-x^2, & y^3-x^3, & \dots, & y^1-x^1 \end{vmatrix}^{1/n}$$

Some properties corresponding to the linear analytic geometry of this metric are developed: quasi-polar coordinates, quasi-direction cosines, quasi-orthogonality. *L. A. Santaló* (Buenos Aires).

★ **Grosche, Günter. Projektive Geometrie. I.** Mathematisch-Naturwissenschaftliche Bibliothek. 7. B. G. Teubner Verlagsgesellschaft, Leipzig, 1957. vi+204 pp. DM 10.20.

Contents of the book: A. Introduction: I. Preliminary sketch. II. Projective space. B. Projective geometry on a straight line. C. Projective geometry on pencils of planes and of lines. D. Projective geometry on fundamental figures of the second kind.

As the contents show, the book is an elementary introduction to Projective Geometry. Projective three-dimensional space is introduced by adding to euclidean space the points at infinity. In Chapter B, the author sets out with the definition of projectivity as a product of perspectivities and proves the fundamental theorem for projectivities among ranges of points by means of the cross-ratio. At the end of Chapter B, Staudt's definition of projectivity is given and is shown equivalent with the previous one, in the case of a real line, by means of continuity considerations. The author states, analytically, the duality law of projective space at the beginning of Chapter C and extends, by duality, the previous results obtained for ranges of points. Projectivity among figures of the second kind is defined also by means of perspectivities in Chapter D. In this Chapter, the theorems of Desargues and Pappus are given by the author. The canonical form for the equations of the projectivities is obtained directly by employing the invariant elements of the projectivity. The book ends with a sketch of the place correlations. Along with the text twenty-one exercises are proposed, with the solutions at the end of the book. — It must be pointed out

that the definition of isomorphism on page 65 is not clear and that the canonical forms given by the author on page 165ff. are affected by the unnecessary parameters b_{ij}. *P. Abellanas* (Madrid).

Lagrange, René. Sur le groupe de la famille des coniques du plan qui ont un élément de contact donné. C. R. Acad. Sci. Paris **244** (1957), 1886–1868.

Let $\{\Gamma\}$ denote the family of the conics in complex projective plane with a given tangent at a given point. Designating this tangent as the infinite line, a cartesian coordinate system x, y can be introduced such that $\{\Gamma\}$ consists of the parabolas Γ: $y=ux^2+vx+w$. The transformations T:

$$x_1=\frac{\alpha x+\beta}{\gamma x+\delta},\ y_1=\frac{\kappa y+ex^2+2fx+g}{(\gamma x+\delta)^2}\ (\alpha\delta-\beta\gamma=1,\ \kappa\neq 0)$$

form a group G which maps $\{\Gamma\}$ onto itself. — Through

$$\mu\colon \Gamma\to\mu\Gamma=(u+w,\ i(u-w),\ v)$$

$\{\Gamma\}$ is mapped one-one onto complex euclidean 3-space E_3. T determines a similarity transformation T^μ: $\mu\Gamma\to\mu T\Gamma$ in E_3 with the similarity factor κ. Through $T\to T^\mu$, G [resp. the subgroup H of the T's with $\kappa=1$] is mapped isomorphically onto the group of similarities [resp. of euclidean motions] in E_3. The last group being generated by the symmetries in E_3, H can similarly be generated by certain "symmetries" which can be described directly in terms of $\{\Gamma\}$. *P. Scherk* (Saskatoon, Sask.).

Mokriščev, K. K. On the trisection of an angle, a segment and a triangle in the Lobačevskiĭ plane. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 103–110. (Russian)

See also: Acel', p. 807; Hatipov, p. 820; MacAdam, p. 847.

Convex Domains, Integral Geometry

Hadwiger, H. Über einen Satz Hellyscher Art. Arch. Math. **7** (1956), 377–379.

Theorem: If each $k+1$ bodies of a denumerable infinity of disjoint, congruent and proper (i.e. with interior points) convex bodies in E_k ($k\geqq 2$) can be cut by a line, then there exists a line which cuts all the bodies of the set. The imposed restrictions are all necessary as is shown by examples. *L. A. Santaló* (Buenos Aires).

Pogorelov, A. V. A general characteristic property of the sphere. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 203–206. (Russian)

A closed convex surface F in E^3 is either a sphere, or a cylindrical zone completed by two hemispheres, or F can be moved into a position F' with the following properties: F and F' have a common point and at that point a common exterior unit normal u_0; moreover there is a positive c and a neighborhood N of u_0 on the unit sphere in which the supporting functions $H(u)$ of F and $H'(u)$ of F' satisfy the inequality $|H(u)-H'(u)|\geqq c|u-u_0|^2$. This theorem yields at once (because the hypothesis of class C^2 eliminates the completed cylinder) the following generalization of a well-known theorem of A. D. Alexandrov: If the principal curvatures k_1, k_2 of a closed convex surface F of class C^2 satisfy a relation $f(k_1, k_2)=$const, where f is a monotone function of both variables, then F is a sphere. *H. Busemann* (Los Angeles, Calif.).

Lenz, Hanfried. Zerlegung ebener Bereiche in konvexe Zellen von möglichst kleinem Durchmesser. Jber. Deutsch. Math. Verein. **58** (1956), Abt. 1, 87–97.

Let B_0 be a plane convex body of diameter 1 and area $|B_0|$. Let $e_m(B_0)$ be the smallest number e such that there exists a covering of B_0 by m convex sets $B_1, \cdots, B_m$ each of diameter at most e. The author obtains the following asymptotic estimates for the value of e_m for large m:

$$\limsup_{m\to\infty} m^{\frac{1}{2}}e_m(B_0)\leqq\sqrt{\frac{8|B_0|}{27^{\frac{1}{2}}}}\approx 1.24\sqrt{|B_0|},$$

$$\liminf_{m\to\infty} m^{\frac{1}{2}}e_m(B_0)\geqq\sqrt{\frac{|B_0|}{\alpha}}>1.19\sqrt{|B_0|},$$

where $\alpha=\max(F_6, \frac{1}{2}(F_5+F_7))$, where F_n is the maximum possible area of an n-gon of diameter 1.

The second half of the paper is devoted to showing that one obtains the same value for e_m if $B_1, \cdots, B_m$ are replaced by arbitrary (not necessarily convex) sets. *D. Gale* (Providence, R.I.).

Differential Geometry

Lane, N. D. Characteristic and order of a differentiable point in conformal n-space. Trans. Roy. Soc. Canada. Sect. III. (3) **50** (1956), 47–52.

In a previous paper the author defined conformal differentiability of points on arcs A in conformal n-space [Pacific J. Math. **6** (1956), 301–313; MR **18**, 668; the notation of that review is assumed]. The n-times differentiable interior points p of A were classified into $(3n-1)\cdot 2^{n-1}$ types. With each of them a characteristic (1) $(a_0, a_1, \cdots, a_n; i)$ was associated with the following properties: (i) $a_\lambda=1$ or 2 ($\lambda=0, 1, \cdots, n-1$); $a_n=1$, 2, or ∞. (ii) $a_0+\cdots+a_k$ is even [odd] if the $(n-1)$-spheres which belong to $\tau_k^{(n-1)}$ but not to $\tau_{k+1}^{(n-1)}$ support [intersect] A at p ($k=0, 1, \cdots, n$); $a_n=\infty$ if $S_n^{(n-1)}$ neither supports nor intersects. (iii) i is the largest integer $\leqq n$ for which $S_i^{(i-1)}=p$.

The (conformal) order of A is the least upper bound of the number of points which A has in common with any $(n-1)$-sphere. The order of a point $p\in A$ is the minimum of the orders of the neighbourhoods of p on A. Let p be an n-times differentiable interior point with the characteristic (1). Then the conformal order of p is greater than or equal to $a_0+a_1+\cdots+a_n$. This theorem is obtained as a corollary of the following one: If the order of p is a finite number, then there exists to every neighbourhood of p an $(n-1)$-sphere arbitrarily close to $S_n^{(n-1)}$ which does not pass through p and which intersects that neighbourhood in $\geqq a_0+\cdots+a_n$ points. The special case $n=2$ of these results was proved by Lane and Scherk [Trans. Amer. Math. Soc. **81** (1956), 358–378; MR **18**, 64]. *P. Scherk* (Saskatoon, Sask.).

Wintner, Aurel. On Frenet's equations. Amer. J. Math. **78** (1956), 349–356.

In the following, X, Y, U_1, $\cdots$ denote vectors in 3-space; U_1, U_2, U_3 are unit vectors. Let (1) Γ: $X=X(s)$ denote an oriented rectifiable Jordan arc with the arc length s. The vector function $Y=Y(s)$ belongs to the class $C^{(n)}$ if $Y^{(n)}=d^nY/ds^n$ exists and is continuous; $\Gamma CC^{(n)}$ if $XCC^{(n)}$. The arc $\Gamma CC''$ belongs to the class F if there is a $U_3=U_3(s)CC'$ normal to $U_1=X'$ such that U_3' is a multiple of the vector product $U_2=[U_3, U_1]$. Let ΓCF. Then (I) Γ has a unique torsion τ, viz. $\tau=\tau(s)=$

$-U_3'U_2=U_3U_2'$; (II) Frenet's formulas hold; (III) $\Gamma\subset C'''$ if and only if the curvature $\kappa(s)$ of Γ is continuously differentiable. {It seems that the definition (2) $\kappa=|X''|$ may conflict with the first Frenet formula $X''=U_1'=\kappa U_2$. Let $X=(t, t^3, t^4,)$, $Y=(2t^3, -2t, 1)$, $U_3=Y/|Y|$, $-1\leq t\leq 1$. Then $\Gamma\subset F$, but (3) $\kappa=U_1'U_2$ will change its sign at $t=0$. If U_3 is continuous, such a change has a geometrical meaning even though sgn $U_1'U_2$ itself has not. As (3) and the continuity of U_3 are essential, (2) and the convention $\kappa\geq 0$ should be avoided.}

Let the surface S be homeomorphic to an open disk. It belongs to the class C^n if it permits a parametric representation S: $X=X(u, v)$ which is n-times continuously differentiable in its region of definition in the (u, v)-plane and for which $[X_u, X_v]$ never vanishes. Let $S\subset C^2$. Then (i) every geodesic on S belongs to F; (ii) a point on S and a tangential direction through it define a unique geodesic torsion; (iii) if the arc $\Gamma\subset S$ belongs to C' and contains no umbilical points, then it is a line of curvature if and only if its geodesic torsion vanishes identically. — Let $S\subset C^3$, $\kappa<0$. Then every asymptotic line on S belongs to F and the Beltrami-Enneper theorem holds.

In an appendix the author proves: Let (1) belong to F. Let $\kappa\neq 0$ for $\lambda=1$, $\tau\neq 0$ for $\lambda=3$. Then the surface R_λ: $X=X(t, s)=t\cdot U_\lambda(s)$ is a torse of class C^2, i.e. the normals of R_λ exist and they are parallel along each generating line $s=$const. This implies that K vanishes on R_λ $(0<|t|<\infty$; $\lambda=1$ or 3). For $\lambda=1$ actually less is assumed.

P. Scherk (Saskatoon, Sask.).

Hsiung, Chuan-Chih. Some global theorems on hypersurfaces. Canad. J. Math. **9** (1957), 5–14.

The author establishes the following theorem which Hopf and Voss obtained for $n=2$ [Arch. Math. **3** (1952), 187–192; MR **14**, 583]: Let V_n, V_n^* be two orientable hypersurfaces twice differentiably imbedded in a Euclidean space E_{n+1} $(n+1\geq 3)$. Suppose that there is a differentiable homeomorphism between V_n, V_n^* such that the orientations are preserved and the line joining every pair of corresponding points P, P^* is parallel to a fixed direction given by the unit vector I, and such that V_n, V_n^* have equal first mean curvatures M_1, M_1^* at every pair of the points P, P^* but no cylindrical elements whose generators are parallel to the fixed direction I. Then V_n, V_n^* can be transformed into each other by a translation. The proof is based on the integral formula

$$n\int_{V_n} w(M_1^*-M_1)I\cdot NdA+\int_{V_n}(1-N\cdot N^*)(dA+dA^*)=0,$$

where w is the distance PP^*; N, N^* the unit normal vectors at P, P^* and dA, dA^* the area elements. From this theorem, other results of Hopf and Voss (paper quoted above) are also extended to $n>2$.

L. A. Santaló (Buenos Aires).

Vincensini, Paul. Sur une interprétation géométrique d'une différentielle totale de la théorie des surfaces minima. Bull. Sci. Math. (2) **79** (1955), 173–180.

Basing his work on some of his earlier investigations [Acad. Roy. Belg. Bull. Cl. Sci. (5) **40** (1954), 1090–1105; MR **16**, 955], the author, in presenting details of proof of his more recently announced result concerning transformations of minimal surfaces [C. R. Acad. Sci. Paris **241** (1955), 153–154; MR **18**, 759], incidentally establishes the reciprocal geometric interpretation referred to in the title of the present paper: It is well known that in rectangular coordinates the equation $z=f(x, y)$ represents a minimal surface if and only if the expression

$$(1+p^2+q^2)^{-\frac{1}{2}}(pdy-qdx)$$

is an exact differential; it is now shown — and can easily be given direct verification — that the equation $z=f(x, y)$ represents a minimal surface if and only if the equation $z'=i\int(1+p^2+q^2)^{-\frac{1}{2}}(pdy-qdx)$, where $i^2=-1$, also represents a minimal surface.

E. F. Beckenbach.

Matsumoto, Makoto. Intrinsic character of minimal hypersurfaces in flat spaces. J. Math. Soc. Japan **9** (1957), 146–157.

When a minimal surface, V^n, is imbedded as a hypersurface of a flat space, it is possible (except in certain circumstances) to obtain algebraic expressions for H_{ij} (the second fundamental tensor) in terms of the curvature tensor. For example, if $R^{ab}H_{ab}=\sigma^{-1}\neq 0$, then $H_{ij}=\sigma S_{ij}$, where $S_{ij}=e(R_{iajb}R^{ab}-R_{ia}R_j^{a})$. Such a surface is said to be of type M^1.

If $\sigma^{-1}=0$, then there exists a similar relationship: $H_{ij}=\sigma_2 S_{2)ij}$ provided $\sigma_2^{-1}\neq 0$. The surface is then said to be of type M^2. By generalization, surfaces of type M^r are defined. For certain surfaces, however, all σ_r^{-1} are zero, and of these (of type M^∞) no such algebraic solution for H_{ij} exists. It is shown that the only possible types for V^n are $M^1, \cdots, M^p$ $(2p+1\leq n)$ and M^∞.

By a change in point of view, the author defines the type M^r for any Riemannian space. Then he proves necessary and sufficient conditions that V^n be a minimal hypersurface of a flat space. Special results are obtained for Einstein spaces, for V^3, and for conformally flat spaces.

C. B. Allendoerfer (Seattle, Wash.).

Alda, Václav. Les transformations isométriques d'un système de hypersurfaces. Czechoslovak Math. J. **6(81)** (1956), 195–211. (Russian. French summary)

A one parameter system $\mathfrak{S}$ of hypersurfaces in n-dimensional euclidean space Σ is mapped onto a similar system $\mathfrak{S}'$ of Σ' in such a way that each hypersurface is transformed isometrically and the orthogonal trajectories are preserved. The author finds two cases can occur: 1) $\mathfrak{S}$ is a system of hyperplanes and the correspondence a one-parameter family of orthogonal transformations; 2) the map is composed of a fixed orthogonal transformation of $n-2$ dimensions together with a map of a two dimensional space depending on one function of two variables. The Pfaffian systems are examined in great detail for $n=3$ and a finer description arrived at.

L. W. Green (Minneapolis, Minn.).

Mayer, O. Familles R de surfaces transversales dans les congruences de droites de l'espace euclidien. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi **2** (1955), 25–42. (Romanian. Russian and French summaries)

Given a congruence of (non-isotropic) straight lines (g):

$$X=x(u, v)+w\lambda(u, v) \quad (\lambda^2=1) \tag{1}$$

in the euclidean 3-space, a 'family R of transversal surfaces' (S) is defined by

$$\frac{\alpha w+\beta}{\gamma w+\delta}=\text{const}, \tag{2}$$

where α, β, γ, δ are uniform functions of u, v and $\alpha\delta-\beta\gamma=\pm 1$. Such a family has, evidently, the following two properties. 1°. Through each point of a straight line (g) passes one and only one surface (S). 2°. Four surfaces

(S) determine on each (g) quadruples of points with equal cross-ratios.

This paper, which is a continuation of an earlier one [Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Ser. I. 5 (1954), no. 3–4, 13–47; MR 16, 1150], studies the invariants of families R of transversal surfaces with respect to the group of transformations which leaves the form of (1) unchanged. The families R are classified from this point of view and geometrical interpretations of the invariants are given.

The paper concludes with a treatment of the dual case, i.e. the straight lines of the congruence are considered the axes of plane pencils and w is replaced by an angular coordinate φ $(-\pi/2<\varphi\leq\pi/2)$. *R. Blum.*

Britan, B. U. Differential geometry of congruences of straight lines and of ruled surfaces of a three-dimensional space of constant curvature. Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 269–278. (Russian)

Les propriétés de la congruence des droites dans un espace R_3 de courbure constante K sont analogues à celles dans l'espace euclidien mais leur traitement exige des considérations plus subtiles vue l'absence du parallélisme absolu.

L'auteur répare la difficulté en introduisant outre la connexion $\{{}_j{}^i{}_k\}_g$ de l'espace donné (avec le tenseur fondamental g_{ij}) une autre connexion $\Gamma^i_{jk}=\{{}_j{}^i{}_k\}_g+S^i_{jk}$ $(i, j, k, l=1, 2, 3)$ dont le tenseur de courbure est $R^i_{jkl}=0$. Pour le tenseur S^i_{jk} on établit la rélation $S^i_{kl}=\omega\varepsilon^i_{kl}$, où $\omega^2=K$, ε^i_{kl} est le tenseur-discriminant formé par le tenseur g_{ij}. La congruence dans l'espace R_3 est définie par les équations

$$x^i=x^i(u^1, u^2, s)$$

$$\frac{\partial^2x^i}{\partial s^2}+\begin{Bmatrix} i \\ k\ l\end{Bmatrix}_g\frac{\partial x^k}{\partial s}\frac{\partial x^l}{\partial s}=0\quad (i, k, l=1, 2, 3).$$

Les vecteurs $V^i=\partial x^i/\partial s$ qui forment le champ des vecteurs tangents satisfont aux équations

$$\frac{\partial D_aV^i}{\partial s}=\omega\varepsilon^i_{kl}V^lD_aV^k\quad (a=1, 2),$$

où

$$D_aV^i=\frac{\partial V_i}{\partial s}+\Gamma^i_{ak}V^a.$$

Ils donnent naissance aux tenseurs fondamenteaux de la congruence

$$\gamma_{ab}=\mathrm{Re}(g_{ij}D_aV^iD_bV^j),\quad \rho_{ab}=\mathrm{Im}(g_{ij}D_aV^iD_bV^j).$$

A l'aide de ces tenseurs on peut exprimer les derivées covariantes des vecteurs V^i, D_aV^i, $D_{ab}V^i$ et montrer la formule analogue à celle de Chasles pour les surfaces réglées de la congruence de l'espace ordinaire. On détermine aussi les surfaces focales et les surfaces centrales de la congruence et enfin on établit à l'aide de méthode connue le théorème d'existence de la congruence donnée par les tenseurs γ_{ab}, $\varphi_{ab}=\gamma_{ab}+\omega\rho_{ab}$ et par les conditions initiales. *F. Vyčichlo* (Prague).

Dolbeault-Lemoine, Simone. Sur la déformabilité des variétés plongées dans un espace de Riemann. Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 357–438.

Let V_{n-1} be a subspace of a locally euclidean V_n. The theorem of Beez shows that if the rank of the second fundamental tensor ≥ 3, V_{n-1} is locally rigid. On the other hand, if this rank is 0 or 1, V_{n-1} is locally euclidean (developable). When the rank is 2, the situation is much more complex. The first portion of this paper investigates the conditions under which subspaces with this rank are deformable in V_n, and classifies the various types of these subspaces.

The classification chosen is that of E. Cartan, as follows: (1) regularly deformable subspaces, namely those for which there is a coordinate system in which the equations governing the second fundamental tensor are the same as those for a $V_2 \subset R_3$; (2) subspaces which are deformable with preservation of an asymptotic direction; (3) subspaces which are deformable with the preservation of distinct conjugate directions. For each of these categories the author obtains necessary and sufficient conditions for the structure of the metric tensor. The results include the existence of subspaces of each type. A similar characterization is given for subspaces deformable with the preservation of the principal directions.

When V_{n-1} is imbedded in a V_n of non-zero constant curvature, the classification remains the same, but few deformable subspaces exist. Specifically, if $n>4$, all subspaces are rigid. When $n=4$, there are no developable subspaces and none deformable with preservation of an asymptotic direction. The remaining cases for $n=4$ are completely characterized.

The last chapter considers certain global properties of deformable subspaces. If a Riemannian V_n $(n>3)$ with positive definite metric contains a complete and deformable V_{n-1}, then V_{n-1} has a non-empty intersection with the set $\Delta(x)$ which is the reciprocal set of each point x of V_n. $\Delta(x)$ is the set containing (a) the points at infinity, if any, on each geodesic through x, and (b) the points ξ such that on a geodesic g through x and ξ, ξ is the first point of the arc $x\xi$ at which the length of g ceases to be equal to the distance from x to ξ. For a locally euclidean V_n, the subspaces V_{n-1} which are both locally deformable and complete are characterized. A complete, locally spherical space of four dimensions contains no V_{n-1} with both these properties.

Finally it is shown that in a euclidean V_n every arcwise connected V_{n-1} which is reducible and complete is the product of the direct sum of $n-p$ lines and a $(p-1)$-dimensional subspace which is arcwise connected, irreducible, and complete. This leads to the result that in such a space the only V_{n-1} which are locally deformable and complete are (1) products $V_2\times R_{n-3}$ and (2) subspaces deformable with the preservation of an asymptotic direction. Hence any V_{n-1} which is complete and locally convex is globally rigid. *C. B. Allendoerfer.*

Bochner, Salomon. Curvature and Betti numbers in real and complex vector bundles. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 225–253.

In a previous paper [Canad. J. Math. 3 (1951), 460–470; MR 14, 90] the author described an N-dimensional vector bundle over an n-dimensional base space M_n by a system of transition matrices $a_A{}^B(y; x)$ between any two intersecting elements (x) and (y) in a fixed covering of M_n by a system of neighborhoods, the $a_A{}^B(y; x)$ being subjected to the consistency properties

$$a_B{}^A(z; y)a_C{}^B(y; x)=a_C{}^A(z; x) \text{ and } a_A{}^B(x; x)=\delta_B{}^A.$$

He calls such a system a matrix structure.

A generic point of the vector bundle is either a contravariant vectoroid u^A with the transition relations $u'^A(y)=a_B{}^A(y;x)u^B(x)$ or a covariant vectoroid t_A with the transition relations $t'_A(y)=a_A{}^B(x; y)t_B(x)$. More generally, if we are

given matrix structures of the form $a_{B_r}{}^{A_r}(y;x)$ ($r=1,2,\cdots,s$) of various dimension N_r, with no connection between them, then their Kronecker product is again a structure. Thus we can define a general mixed tensoroid of the type $t^{A_1A_2\cdots A_p}{}_{B_1B_2\cdots B_q}$. We need hardly say that ordinary tensors are defined with respect to the classical structure $\partial y^i/\partial x^j$.

In the first two sections of the present paper, the author develops the tensor algebra and tensor analysis in such metric or non-metric bundles. In section 3, the author proves the following. (I) On a compact metric M_n, if a vectoroid ξ_A satisfies an equation of the form

$$g^{rs}\xi_{A,r,s}=S_{AB}g^{BC}\xi_C,$$

where the comma denotes covariant differentiation leaving the metric invariant, and if the tensoroid S_{AB} is positive definite, then ξ_A is identically zero. If S_{AB} is only positive semi-definite, then $\xi_{A,r}=0$ and $S_A{}^C\xi_C=0$. (II) On a non-compact M_n, the entire conclusion holds if the vectoroid has "boundary value zero" in the following sense. To each $\varepsilon>0$ there corresponds a compact subset $M_n{}^\varepsilon$ of M_n such that $g^{AB}\xi_A\xi_B<\varepsilon^2$ for all points outside $M_n{}^\varepsilon$.

In section 4, the author considers a symmetric or skew-symmetric tensoroid of the type $\xi_{A_1A_2\cdots A_sB}$ in a compact M_n which satisfies differential equations of the type

$$g^{rs}\xi_{A_1A_2\cdots A_sB,r,s}=S_{\dot A_1\dot A_2\cdots\dot A_s}{}^{C_1C_2\cdots C_s}{}_{\dot B}{}^D\xi_{C_1C_2\cdots C_sD},$$

where the tensoroid S satisfies

$$T=S^{A_1A_2\cdots A_sC_1C_2\cdots C_sBD}\xi_{A_1A_2\cdots A_sB}\xi_{C_1C_2\cdots C_sD}=$$
$$sP_{\dot A}{}^C{}_{\dot B}{}^D\xi^{AA_2\cdots A_sB}\xi_{CA_2\cdots A_sD}+Q_{\dot B}{}^D\xi^{A_1A_2\cdots A_sB}\xi_{A_1A_2\cdots A_sD}.$$

Then in consequence of the positivity assumption $P_{\dot A}{}^C{}_{\dot B}{}^Dt^{AB}t_{CD}>0$, whenever $t^{AB}t_{AB}>0$, there is an integer s_0 such that for $s\geqq s_0$, the tensoroid $\xi_{A_1A_2\cdots A_sB}$ is identically zero. The author next applies the above theorem to the "harmonic" tensoroid $\xi_{Ai_1i_2\cdots i_p}$ defined as a tensoroid satisfying

$$\xi_{Ai_1i_2\cdots i_p,j}=\sum_{q=1}^{p}\xi_{Ai_1i_2\cdots j\cdots i_p,i_q}\text{ and }g^{ij}\xi_{Ai_1i_2\cdots i_{p-1}i,j}=0.$$

The last section is devoted to complex vector bundles. Whether the base space M is real or complex, we introduce a matrix structure $a_B{}^A(y;x)$ of dimension N in which the coefficients are complex-valued. Vectoroids u^A, t_A have components which are complex-valued too. With any complex structure $\{a_B{}^A(y;x)\}$, we can associate the conjugate complex structure $\{a_{B^*}{}^{A^*}(y;x)\}$, where

$$a_{B^*}{}^{A^*}(y;x)=\overline{a_B{}^A(y;x)},$$

and also the logical sum of the two stuctures

$$a_{\mathfrak{B}}{}^{\mathfrak{A}}(y;x)=\begin{Bmatrix} a_B{}^A(y;x) & 0\\ 0 & a_{B^*}{}^{A^*}(y;x)\end{Bmatrix}.$$

If t_A is a vectoroid for $a_B{}^A(y;x)$ then $t_{A^*}=\bar t_A$ is a vectoroid for $a_{B^*}{}^{A^*}(y;x)$ and the object $t_{\mathfrak{A}}=\{t_A,t_{A^*}\}$ is a vectoroid for their logical sum.

If the base space is complex and $L_{B\mathfrak{i}}{}^A$ is an affine connection pertaining to $a_B{}^A(y;x)$, then the enlarged object $L_{\mathfrak{B}\mathfrak{i}}{}^{\mathfrak{A}}$ with the additional components $L_{B^*\mathfrak{i}}{}^{A^*}=\bar L_{B\mathfrak{i}}{}^A$, $L_{B^*\mathfrak{i}}{}^A=L_{B\mathfrak{i}}{}^{A^*}=0$ is an affine connection for $a_{\mathfrak{B}}{}^{\mathfrak{A}}(y;x)$. The fundamental tensor $g_{\mathfrak{A}\mathfrak{B}}$ pertaining to $a_{\mathfrak{B}}{}^{\mathfrak{A}}(y;x)$ shall be of the kind $g_{\mathfrak{A}\mathfrak{B}}=g_{\mathfrak{B}\mathfrak{A}}$, $g_{AB}=g_{A^*B^*}=0$, $g_{A^*B}=\bar g_{AB^*}$, $g_{AB^*}t^At^{B^*}>0$ for $t^At^{A^*}>0$. We demand of course $g_{\mathfrak{A}\mathfrak{B},\mathfrak{i}}=0$. Then the quantity $L=g^{\rho\sigma^*}g^{AB^*}g^{\alpha\beta^*}L_{A\alpha\sigma^*}{}^Ct_{C\rho}\bar t_{B^*\beta^*}$ is real valued.

If the index A refers to a complex structure $a_B{}^A(y;x)$ and $\alpha_1,\cdots,\alpha_p$ to the "unstarred" parameters in $x^i=(x^\alpha,x^{\alpha^*})$ then the "partial" tensoroid $t_{A\alpha_1\alpha_2\cdots\alpha_p}$ shall be called harmonic if it is antisymmetric in $\alpha_1,\cdots,\alpha_p$ and $t_{A\alpha_1\cdots\alpha_p,\rho}=\sum_{q=1}^p t_{A\alpha_1\cdots\alpha_{q-1}\rho\alpha_{q+1}\cdots\alpha_p,\alpha_q}$, and $g^{\sigma\rho^*}t_{A\alpha_1\cdots\alpha_{p-1}\rho,\sigma^*}=0$. The author proves: If in a compact Kähler base space there is given a principal complex structure $a_B{}^A(y;x)$ and some other secondary complex structure $b_D{}^C(y;x)$ both with linear connections and fundamental tensoroids, and if the affine connection of the principal structure is positive in the sense that the form L is positive definite, then for sufficiently large s, any tensoroid $\xi_{A_1A_2\cdots A_sB\alpha_1\alpha_2\cdots\alpha_p}$ which is harmonic relative to the space indices $\alpha_1,\alpha_2,\cdots,\alpha_p$ in the manner stated above must be zero. Here each index A_r refers to the principal structure and B to the secondary structure, and $\xi_{A_1A_2\cdots A_sB\alpha_1\alpha_2\cdots\alpha_p}$ is assumed either symmetric or antisymmetric in the indices $A_1,A_2,\cdots,A_s$.

K. Yano.

Yaglom, I. M. Curves in symplectic space. Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 119–137. (Russian)

Frenet formulas for curves in projective symplectic space were obtained by Chern and Wang [Sci. Rep. Nat. Tsing Hua Univ. **4** (1947), 453–477; MR **10**, 65]. The author constructs a similar theory of curves in affine symplectic spaces. Curvatures and torsions are defined for a "general" point of a "general" curve (all the contact elements span the space) in terms of a natural parameter, the symplectic arc length. These functions determine the curve. (The curvatures, however, are not independent.) The differential equations are integrated explicitly for a special case of constant curvatures and torsions.

L. W. Green (Minneapolis, Minn.).

Hatipov, A. È.-A. Theory of surfaces in a space with a decomposed absolute. Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 285–308. (Russian)

Die Formeln der Flächentheorie in den üblichen nicht-euklidischen Räumen sind schon seit langem bekannt und finden sich im Band 2 des Bianchischen Lehrbuchs dargestellt. In Ergänzung hierzu gibt Verf. die entsprechenden Formeln für einen reellen Raum P_3, in dem ein Paar konjugiert komplexer Ebenen w_1 und w_2 als absolutes Gebilde ausgezeichnet ist. In geeigneten Koordinaten ist mithin die quadratische Form $x_1^2+x_2^2$ vorgegeben, sodaß w_1 und w_2 die Gleichungen $x_1\pm ix_2=0$ haben. Ihre Schnittgerade $x_1=x_2=0$ heißt die absolute Achse. Zwischen 2 reellen Punkten allgemeiner Lage und 2 Ebenen durch die Achse kann man dann in üblicher Weise durch den log des Doppelverhältnisses Abstände und Winkel erklären. Durch die Normierung $x_1^2+x_2^2=1$ werden für Punkte ausserhalb der Achse Weierstrasskoordinaten wie in den nicht entarteten Geometrien eingeführt. Darauf wird dann der gesamte Formelapparat der inneren und äußeren Differentialgeometrie einer Fläche F in diesem Raum entwickelt, wobei F stets gleichzeitig als Punkt- und Tangentialgebilde betrachtet wird. Einem allgemeinen Punkt $P\epsilon F$ läßt sich eine beliebige, die Achse schneidende Gerade durch P als Normale 1. Art zuordnen, während die Normale 2. Art als Schnitt der Tangentialebene von P mit der zu P bezüglich des absoluten Gebildes polaren Ebenen i. a. eindeutig erklärt ist. Die Unbestimmtheit in der Erklärung der Normalen 1. Art läßt sich dadurch beseitigen, daß man auf der Achse noch 2 konjugiert komplexe Punkte auszeichnet und so zu einer in sich dualen Geometrie gelangt. Hier gewinnt der Verf. enge Beziehungen zu früheren Arbeiten von Norden [dieselben Trudy **6** (1949), 125–224; MR **15**, 61]. Nach Norden sind mit den beiden Arten von

Normalen 2 affine Übertragungen auf der Fläche erklärt, die im vorliegenden Falle beide äquiaffin zu nennen sind. Die abwickelbaren Flächen haben jetzt bei richtiger Normierung das Krümmungsmaß 1 des umgebenden Raumes. Gegen Schluß wird noch die Klasse der Drehflächen um die absolute Achse gekennzeichnet. Es sind dies Flächen mit einer inneren projektiv-euklidischen Geometrie nach einer ebenfalls von Norden stammenden Bezeichnung [Mat. Sb. N.S. **18(60)** (1946), 139–152; MR **8**, 94]. *W. Burau* (Hamburg.)

See also: Bögel, p. 793; Obata, p. 822.

Riemannian Geometry, Connections

Newns, W. F.; and Walker, A. G. Tangent planes to a differentiable manifold. J. London Math. Soc. **31** (1956), 400–407.

On sait que Chevalley a défini [Theory of Lie groups, v. 1, Princeton, 1946; MR **7**, 412] les vecteurs tangents en un point d'une variété analytique comme les fonctionnelles linéaires t, définies sur l'espace des fonctions locales et qui satisfont à la règle

$$t(fg)=tf\cdot g(p)+f(p)\cdot tg. \tag{1}$$

Les auteurs montrent que cette définition reste valable pour les variétés de classe infinie. Ils montrent, au contraire, que l'ensemble des fonctionnelles linéaires t satisfaisant à (1) forment un espace vectoriel de dimension égale à la puissance du continu dès que la classe de la variété cesse d'être infinie. Certains des résultats précédents avaient effective ment été établis par le rapporteur, C. R. Acad. Sci. Paris, **241** (1955), 19–20; **242** (1956), 1573–1575; MR **16**, 1152; **17**, 892.] *G. Papy* (Bruxelles).

Nožička, František. On imbeddings of total geodesics in a Riemannian space. I, II. Časopis Pěst. Mat. **78** (1953), 65–72, 215–228. (Czech)

La variété à $n-1$ dimensions dans l'espace de Riemann V_n à n dimensions est totalement géodésique quand pour le second tenseur fondamental on a $h_{ab}\equiv 0$ $(a, b=1, \cdots, n-1)$. Le vecteur principal v^α de la variété V_n satisfait à l'équation

$$(h_{\alpha\beta}-\sigma g_{\alpha\beta})v^\beta=0 \quad (\alpha, \beta=1, \cdots, n),$$

où $g_{\alpha\beta}$ est le premier et $h_{\alpha\beta}$ le second tenseur de V_n. On démontre le théorème bien connu: S'il existe dans $V_n \subset E_{n+1}$, où E_{n+1} est l'espace euclidien à $n+1$ dimensions, la variété g_{n-1} à $(n-1)$ dimensions totalement géodésique, alors le vecteur normal de g_{n-1} en chaque point de g_{n-1} a la direction du vecteur principal de la variété V_n en point pris en considération. On généralise les considérations pour les variétés totalement géodésiques à $p<n-1$ dimensions dans l'espace de Riemann $V_n\subset E_{n+1}$. On obtient le résultat: Soit P_0 un point de l'espace V_n et π_0 un p vecteur de V_n en point P_0. S'il existe une variété $g_p\subset V_n$ à p dimensions, totalement géodésique, qui passe par le point P_0 et contient en point P_0 le p-vecteur π_0 et qui n'est pas le hyperplan à p dimensions dans V_n, alors: 1) g_p est l'unique variété totalement géodésique à p dimensions dans V_n et 2) g_p est située dans l'éspace linéaire E_{p+1} à $p+1$ dimensions qui contient le point P_0, le p-vecteur π_0 en P_0 et aussi le vecteur normal de l'espace V_n en P_0. *F. Vyčichlo* (Praha).

Chandra Chaki, Manindra. Some theorems on recurrent and Ricci-recurrent spaces. Rend. Sem. Mat. Univ. Padova **26** (1956), 168–176.

This paper is concerned with the (Riemannian) recurrent spaces K_n of H. S. Ruse, i.e. those whose curvature tensor satisfies $R_{ijkl,m}=R_{ijkl}\lambda_m$ for some λ_m, and the Ricci-recurrent spaces R_n of E. M. Patterson, i.e. those whose Ricci tensor satisfies $R_{ij,k}=R_{ij}\lambda_k$, $R_{ij}\not\equiv 0$. Conditions are found for a space to be conformal to a K_n or to an R_n. *A. G. Walker* (Liverpool).

★**Nomizu, Katsumi. Lie groups and differential geometry.** The Mathematical Society of Japan, 1956. xiv+80 pp.

This booklet achieves in an extremely satisfactory manner its purpose of giving to those already familiar with some rudimentary concepts of differential geometry, a formulation of the most important properties of connections, in 'intrinsic' language. It is specially suitable as a text for a course in the subject.

The first chapter starts with the definition of a differentiable manifold as a Hausdorff space on which a ring of C^∞ functions is given abstractly. Within a few pages, the concepts of tangent vector, vector field, bracket operator, (local) one-parameter group are defined, and the most important relations are established. The author proceeds at the same rate through differential forms, tensors, transformation groups, homogeneous spaces and fiber bundles. It is clear that this chapter is meant only as a refresher, for several proofs (or details of proofs) are left to the reader.

Chapter II deals with connections in fiber bundles, essentially following Ehresmann's approach [Colloque de Topologie (espaces fibrés), Bruxelles, 1950, Thone, Liège, 1951, pp. 29–55; MR **13**, 159] of horizontal planes in the principal bundle, and the alternative of Lie algebra valued differential forms. Holonomy groups are defined as subgroups of the structure group. The holonomy theorem, establishing the Lie algebra of the restricted holonomy group in terms of the curvature form, is proved by reducing the structure group of the bundle to the holonomy group, and then constructing a distribution of planes in the bundle space, each of whose integral manifolds must contain all points joinable to a given one of its points by a horizontal path. The argument is a good bit simpler than either that of Ambrose and Singer [Trans. Amer. Math. Soc. **75** (1953), 428–443; MR **16**, 172], or Nijenhuis [Nederl. Akad. Wetensch. Proc. Ser. A. **56** (1953), 233–240, 241–249; **57** (1954), 17–25; MR **16**, 171, 172]. The chapter is concluded with a discussion of local flatness, existence of connections, and connections in associated bundles.

The third and last chapter deals with the classical case of connections in the bundle of frames, and bundles associated to it. Besides the curvature, also the torsion arises now, and the situation is studied in some detail. Not until now do the classical $\Gamma_{ij}{}^k$ enter the discussion, and relatively little use is made of them. Of course, we find Bianchi's identity, geodesics and completeness; normal coordinates and Killing vector fields. Then come spaces with $\nabla T=0$, $\nabla R=0$; and the affine connection (Cartan connection) is briefly mentioned in the last section.

The set-up makes it obvious that the author prefers to avoid index notations. He shows clearly how, at least in the material he discusses, the 'intrinsic' methods are elegant and adequate. To what extent this would remain

true in more complicated problems, involving higher order covariant derivatives and such, remains to be seen, of course.

After reading this booklet the reader will find himself adequately prepared to study Riemannian spaces (only mentioned in a footnote), and such subjects as Hermitian, almost complex and Kählerian stuctures.

A. Nijenhuis (Seattle, Wash.).

Vrănceanu, G. Sur les espaces à connexion affine localement euclidiens. Publ. Math. Debrecen **4** (1956), 359–361.

It is a well-known result of E. Cartan [Leçons sur la géométrie des espaces de Riemann, 2e éd., Gauthier-Villars, Paris, 1951, p. 62; MR **13**, 491] that if a Riemann space V_n is simply connected, locally euclidean and, moreover, complete (in the sense that any infinite bounded sequence has a limit point), then it is globally equivalent to a euclidean space S_n.

The author first extends this result to a simply connected, affinely connected manifold, A_n, which is locally flat: namely, he proves that on such a manifold cartesian coordinates may be (globally) introduced, so that A_n is mapped (uniquely) into a euclidean space E_n. (In general, however, the mapping is neither one-to-one, nor onto.)

Assume now the affine connexion is induced by a positive definite Riemann metric — and name the space V_n—: the above mapping is then distance-preserving, and therefore must be one-to-one.

From this, the author derives a more direct proof of Cartan's result, under the further assumption that the space be complete. Indeed, suppose there is $P \in E_n$, which is not the image of a point of V_n, and consider a point $P_0 \in E_n$, whose inverse image is $M_0 \in V_n$. On the line P_0P there is a point Q which divides the points of P_0P that are images of points in V_n from the points that are not. Consider now a sequence $P_i \in P_0Q$, having Q as limit: their inverse images $M_i \in V_n$ form obviously an infinite bounded sequence, and therefore have a limit point M, the inverse image of Q. Therefore there must be a neighborhood of M which is mapped onto a neighborhood of Q, which contradicts the assumption that on the line segment QP there are no points images of points of V_n.

V. Dalla Volta (Rome).

See also: Craig, p. 798; Laugwitz, p. 809; Matsumato, p. 818; Britan, p. 819; Dolbeault-Lemoine, p. 819.

Complex Manifolds

Obata, Morio. Affine transformations in an almost complex manifold with a natural affine connection. J. Math. Soc. Japan **8** (1956), 345–362.

A. Lichnerowicz [C. R. Acad. Sci. Paris **239** (1954), 1344–1346; MR **17**, 531], J. A. Schouten and K. Yano [Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 565–570; MR **17**, 531] proved that, in an irreducible (pseudo-) Kählerian manifold of dimension $2m$, if m is odd or if m is even and the Ricci curvature tensor does not vanish, then the largest connected group of isometries preserves the (almost) complex structure. The author of the paper under review tries to generalise the above result.

After some algebraic preliminaries, he proves the following theorems. (I) Let M be an almost complex manifold of dimension $2m$ with the almost complex structure F. We denote by $\mathfrak{H}_p$, $p \in M$, the homogeneous holonomy group of M with respect to a natural affine connection (i.e. an affine connection ∇ such that $\nabla F=0$). $A(M)$ denotes the group of all affine transformations of M onto itself and $A_0(M)$ denotes the connected component of the identity of $A(M)$. We assume that $\mathfrak{H}_p$ is irreducible (in R: field of real numbers). Then $\mathfrak{H}_p$ is a subgroup of $\mathrm{CL}(m, R)$ (i.e. real representation of $\mathrm{L}(m, C)$, $\mathrm{L}(m, C)$ being the group of all regular matrices of degree m with coefficients in C.) Further, (1) in case m is odd, or m is even ($m=2l$) and $\mathfrak{H}_p$ is not a subgroup of $\mathrm{QL}(l, R)$ (i.e. the quaternionian linear group), we have $\varphi \cdot F=\pm F$ for every $\varphi \in A(M)$. Especially, we have $\varphi \cdot F=F$ for every $\varphi \in A_0(M)$, i.e. $A_0(M)$ preserves the almost complex structure. (2) In case m is even ($m=2l$) and $\mathfrak{H}_p$ is a subgroup of $\mathrm{QL}(l, R)$, M has three independent almost complex structures F, G and H such that $FG=-GF=H$, $GH=-HG=F$, $HF=-FH=G$ and they are all parallel. $A(M)$ acts on the vector space spanned by F, G and H as a group of orthogonal transformations. Furthermore these orthogonal transformations belong to SO(3) in the vector space.

(II) In an irreducible pseudo-Kählerian manifold M of dimension $2m$, if m is odd, or if $m=2l$ and $\mathfrak{H}_p$ is not a subgroup of the real representation of the unitary symplectic group, then $A_0(M)$ preserves the almost complex structure.

(III) In an irreducible complex manifold of dimension $2m$, if m is odd, or if $m=2l$ and the homogeneous holonomy group is not a subgroup of $\mathrm{QL}(l, R)$, an infinitesimal affine transformation is always complex analytic.

The result of A. Lichnerowicz, J. A. Schouten and K. Yano mentioned at the beginning is a corollary to theorems (II) and (III).

K. Yano (Tokyo).

See also: Bochner, p. 819.

Algebraic Geometry

Spampinato, Nicolò. La superficie approssimante una falda di Halphen nell'n-mo ordine del suo punto origine. Ricerche Mat. **5** (1956), 226–238.

Farchi, Vittorio. Le curve degeneri del sistema algebrico delle parabole dei minimi quadrati. Rend. Mat. e Appl. (5) **15** (1956), 291–314 (1957).

Anonymous. Correspondence. Amer. J. Math. **78** (1956), 898.

Le classique théorème de Chow (toute sous-variété analytique V d'un espace projectif complexe est algébrique) peut être rapidement démontré dans le cas où V est sans singularité, en utilisant les deux résultats suivants: (a) si V est irréductible et de dimension n, le corps des fonctions méromorphes sur V a un degré de transcendance au plus égal à n; (b) si W est une sous-variété algébrique et algébriquement irréductible, W est analytiquement irréductible.

H. Cartan (Paris).

Fáry, István. Cohomologie des variétés algébriques. Ann. of Math. (2) **65** (1957), 21–73.

In einer früheren Arbeit [Ann. of Math. (2) **63** (1956), 437–490; MR **17**, 1118] hat Verf. einer stetigen Abbildung $f: X \to Y$ eine Spektralsequenz zugeordnet, die in dieser Arbeit für den Fall untersucht wird, wo X eine komplexe Mannigfaltigkeit, Y die Ebene C der komplexen Zahlen

und f eine in X definierte holomorphe Funktion ist. Von f wird dabei weiter vorausgesetzt, daß $f(X)=C$, daß es in X nur endlich viele "d-kritische" Punkte $c_1, \cdots, c_\mu$ gibt, in denen df verschwindet, daß $f(c_i)\neq f(c_j)$ für $c_i\neq c_j$ und daß es zu jedem $\xi \in C$ eine offene Menge U von X gibt, mit folgenden Eigenschaften: 1) U enthält das Komplement einer kompakten Teilmenge von $f^{-1}(\xi)$. 2) U ist topologisch-trivial gefasert über einer offenen Teilmenge D von C mit f als Projektion und $f^{-1}(\xi)\cap U$ als typischer Faser. Man setze $\Gamma=\{f(c_1), f(c_2), \cdots, f(c_\mu)\}$. Die zu f gehörige Spektralsequenz hat dann folgende Struktur.

$$E_2= \sum_{\gamma\in\Gamma} H^*(f^{-1}(\gamma))+H^1(C-\Gamma, \mathscr{F}_0)+H^2(C-\Gamma, \mathscr{F}_0),$$

wo $\mathscr{F}_0$ das lokale Koeffizientensystem $H^*(f^{-1}(\xi))$ ist ($\xi \in C-\Gamma$). Ferner ist $E_4=E_5=\cdots=E_\infty$ und

$$E_\infty = \sum_{i=-1}^{2} H^*(X)_i/H^*(X)_{i+1}$$

bezüglich einer Filtrierung

$$H^*(X)=H^*(X)_{-1}\supset H^*(X)_0= H^*(X)_1\supset H^*(X)_2\supset H^*(X)_3=0.$$

In der umfangreichen und interessanten Arbeit werden nun Anwendungen dieser Spektralsequenz auf die Cohomologie (affiner und projektiver) algebraischer Mannigfaltigkeiten gegeben. Es ergeben sich u.a. Beweise für berühmte Sätze von Lefschetz [L'analysis situs et la géométrie algébrique, Gauthier-Villars, Paris, 1924; Géometrie sur les surfaces et les variétés algébriques, ibid., 1929]. Erwähnt werde hier nur der Lefschetzsche Satz über die Hyperebenenschnitte, den Verf. für die ganzzahlige Cohomologie beweist, und ferner noch die Lefschetzsche Theorie der "verschwindenden Zyklen". Weiter ergibt sich mit Hilfe der Lefschetzschen Theoreme durch Rekursion eine vollständige Bestimmung der additiven Cohomologiestruktur der "vollständigen Durchschnitte" im affinen bzw. projektiven Raum. Verf. weist auf Untersuchungen des Ref. hin, der die vollständigen Durchschnitte ebenfalls behandelt hat [Proc. Internat. Congress Math., Amsterdam, 1954, v. III, Noordhoff, Groningen, 1956, pp. 457–473]. *F. Hirzebruch* (Bonn).

Igusa, Jun-ichi. On a problem of Picard concerning symmetric compositums of function-fields. Illinois J. Math. 1 (1957), 105–107.

Let Σ be the function-field of a variety V^r over an algebraically closed field K of arbitrary characteristic; let $\Sigma(m)$ be the invariant subfield of the m-fold direct compositum of Σ under the symmetric group G of permutations of factors. If $r, m>1$ then it is shown that $\Sigma(m)$ can never become an abelian function-field, i.e. the function-field of an abelian variety. For $r=2$ this fact was observed by Picard. (Clearly, $\Sigma(1)$ is abelian if and only if Σ is abelian, and it is well known that, for $r=1$, $\Sigma(m)$ is abelian if and only if m is the genus of Σ.)

The proof goes as follows. Let U be the m-fold direct product of V. Let $\Omega=\sum_{q=0}^{\infty}\Omega_q$ denote in general the graded algebra over K of differential forms of the first kind on a variety and let $\Omega^G(U)$ be the G-invariant differential forms of the first kind on U. By a general theorem there is a bijective K-linear isomorphism of the m-fold tensor product of $\Omega(V)$ over K to $\Omega(U)$. Now assuming that $\Sigma(m)$ is the function-field of an abelian variety A, there is also a bijective K-linear isomorphism of $\Omega(A)$ to $\Omega^G(U)$; furthermore, as is well-known $\dim_K \Omega_1(A)=\dim A=rm$.

Using these facts one obtains formally a contradiction, in case $r, m>1$, by comparison of the $\dim_K \Omega_i(V)$ and $\dim_K \Omega_i^G(U)$ for $i=1, 2, \cdots, m+2$. *J. P. Murre.*

★ **Cartan, Henri. Quotient d'un espace analytique par un groupe d'automorphismes.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 90–102. Princeton University Press, Princeton, N. J. 1957. $7.50

The author here presents results in connection with certain important aspects of the following problem: Given a general analytic space X and a suitably discontinuous group G of analytic automorphisms of X, what can be said about the structure of the orbit space X/G? (The definition of general analytic space is given on p. 96 of this article.) It is first shown that if K is a commutative, algebraically closed field, and if G is a finite group of linear automorphisms of K^n, then K^n/G may be viewed as a normal affine algebraic variety imbedded in K^q, for some suitable q (q is not unique). In case K is the complex numbers C, C^n/G is an algebraic variety V imbedded in C^q, for which, given any $x \in C^n$, the local ring of V at $\varphi(x)$, the canonical image of x in V, is isomorphic to the ring $H^{G(x)}$ of holomorphic functions invariant under $G(x)$ at x, where $G(x)$ is the subgroup of G leaving x fixed. One of the theorems used in the proof of this fact states that $H^{G(x)}$ is analytically generated by a finite number of homogeneous polynomials belonging to $H^{G(x)}$. In fact, if $Q_1, \cdots, Q_q$ are homogeneous polynomials generating the ring of polynomials invariant under $G(x)$, and if H' is the ring analytically generated by $Q_1, \cdots, Q_q$, it is shown that H' is closed in $H^{G(x)}$ in the topology of the maximal ideal, and since it is clear that H' is dense in $H^{G(x)}$, $H'=H^{G(x)}$. Since a finite group of complex automorphisms can be made locally linear by a suitable choice of local coordinates (constructed by an averaging process), it then follows, by simple considerations, that if X is a complex analytic manifold and G a suitably discontinuous group of automorphisms of X, X/G is a general analytic space. If, more generally, X is a general analytic space, X/G is still a general analytic space, and the proof of this is reduced to the previous case by Lemma 2, p. 98. The article concludes by proving that if X is a bounded domain in C^n and if X/G is compact, then X/G may be imbedded as a projective algebraic variety in a complex projective space, the coordinates of the imbedding being Poincaré series of a suitable high weight. The result is stated in broader terms, but the example of a bounded domain is a prototype for the more general conclusions. *W. L. Baily* (Cambridge, Mass.).

NUMERICAL ANALYSIS

★ **Lanczos, Cornelius. Applied analysis.** Prentice Hall, Inc., Englewood Cliffs, N. J., 1956. xx+539 pp. $9.00.

This book provides stimulating treatments of a large number of aspects of "parexic analysis", to use the term suggested by the author, or of "numerical analysis" and its relatives, in the more usual parlance. It is written in a pleasing, almost conversational style, with each chapter prefaced by interesting historical notes. Together with fresh treatments of many classical topics, there is a liberal sprinkling of recently developed material, much of which is directly attributable to the author himself.

The beginning student may be troubled somewhat by a certain unevenness of pace and by a rather frequent use in one chapter of methods to be developed (or of results to be established) in a later chapter. But such points detract little from the value of the work as a source not only of a variety of useful techniques and procedures but also of an insight into the sort of reasoning which leads to their invention.

The first chapter deals rather briefly with the solution of algebraic equations. In addition to special devices applicable to cubics and quartics, the methods of Newton and of Bernoulli are presented and a "scanning" method for dealing with complex roots of nearly equal magnitude is also included.

Chapters two and three treat properties of matrices, principally with respect to eigenvalue problems, with iterative methods of solving large systems of linear equations introduced rather as a by-product.

The fourth chapter deals enthusiastically and engagingly with techniques and applications of harmonic analysis. It contains a particularly commendable treatment of the Gibbs effect, and of methods for counteracting it, as well as a group of novel methods for the approximate inversion of Laplace transforms.

Chapter five, entitled "Data Analysis", first deals very briefly with polynomial interpolation and differentiation, then, in somewhat more detail, with least-square polynomial and trigonometric approximations and related methods of smoothing empirical functions.

In chapter six the quadrature formulas of Simpson and Gauss are treated, together with some other methods of numerical integration. The author includes several ingenious methods for estimating associated error bounds, together with warnings that these methods are not universally reliable.

The final chapter deals principally with the use of Chebyshev polynomials in economizing polynomial approximations and with the author's so-called "τ method" for generating polynomial (or rational) approximations to solutions of linear ordinary differential equations with polynomial coefficients.

Appropriately selected numerical examples are interspersed throughout the text and several useful numerical tables are presented in an appendix. No problems or exercises are included.

This book unquestionably belongs on the reference shelf of any serious worker in the field of computational analysis. *F. B. Hildebrand* (Cambridge, Mass.).

Numerical Methods

★ **Interpolation and allied tables. Prepared by H. M. Nautical Almanac Office.** Her Majesty's Stationery Office, London, 1956. 80 pp. 5 shillings.

This is new edition of a booklet with the same title, first issued in 1936 as a reprint of certain pages of the British Nautical Almanac, and which was frequently reprinted, sometimes with additions. In its present form it provides an excellent introduction to a large part of classical numerical analysis and can be thoroughly recommended as a compact text, complete with tables.

It is in three parts. Part I contains a section on elementary computing principles, which is followed by an account of the principles and practices of interpolation and some remarks on numerical differentiation and integration. There are worked examples showing the use of the tables in the second part.

Part II contains various tables of Bessel and Everett coefficients, some critical and some direct. The main tables give B_2, B_3, B_4 to 6D (with differences) and E_2 and F_2 (to 7D), E_4 and F_4 (to 6D), E_6 and F_6 (to 5D), together with the residual coefficients M_4 and N_4 (to 3D) and T_4 (to 2D); in each case the argument range is $p=0(.001)1$.

Part III is a collection of relevant formulas, subdivided into eight sections, some of which, e.g., that on the solution of ordinary differential equations, contain much valuable commentary.

The titles of the sections are: Operators; Differences; Interpolation; Differentiation; Integration; Differential equations; Precision of calculation; Constants. There is also a bibliography.

The problem of sub-tabulation is to be dealt with in a companion booklet. The typography is excellent and in this respect the booklet provides an example for table-makers. *John Todd* (Washington, D.C.).

Heyda, P. G. A simple numerical example for the beginner of Kron's method of tearing and interconnecting. Matrix Tensor Quart. 6 (1956), 142–145.

A numerical example of Kron's method applied to a set of rods in an elastic webbing is given. *C. Saltzer.*

★ **Hestenes, Magnus R. The conjugate-gradient method for solving linear systems.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 83–102. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

During the last ten years some methods of solving a large linear system of equations

$$Mx=k$$

have been developed, characterized by the property of finite iteration. Let x_i be the ith approximation of the solution h. Then a direction vector p_i is chosen and the next approximation computed by

$$x_{i+1}=x_i+a_ip_i.$$

The scalar a_i is determined by the condition that an appropriate error-measure is minimized during this step of iteration. Such an iteration is said to be finite, if after a finite number of steps the last approximation coincides with the solution of the given system. The basic problem is of course to construct a set of direction vectors p_i in such a way that the iteration becomes finite. In 1952 a finite iteration called "method of conjugate gradients" was discussed in a joint paper of M. Hestenes and the reviewer [J. Res. Nat. Bur. Standards **49** (1952), 409–436; MR **15**, 651].

In the paper under consideration the author outlines a general theory of finite iterations. Generalising the "method of conjugate directions" (established already in the earlier paper) he gives an algorithm for generating appropriate direction vectors p_i and proves various properties of this construction. Later on the general method of conjugate directions is specialised to a generalized method of conjugate gradients (cg-method). In this section the reader will find the following interesting result. Each finite iteration is a cg-iteration in the generalized sense of the paper. Hence from a theoretical point of view the dicussion about finite iterations is closed. In

the last part of the paper different computational routines are worked out corresponding to various choices of the error-measure. One of these algorithms must be emphasized because it minimizes in each step the Euclidian distance $|x_i-h|$ from the solution point h of the given system. It is very surprising that the author was able to establish such an iteration because the quantity $|x_i-h|$ cannot be computed until the solution h has been obtained. There is moreover a modification of cg-iteration minimizing the sum of the squared residuals in each step. cg-methods are sensitive to rounding-off errors. They have however their merits for dealing with ill-conditioned systems as occur if partial differential equations are solved by difference methods. In this case one may liquidate most of the residuals by some elementary iteration and then improve the approximation obtained by a few steps of cg-algorithm. *E. Stiefel* (Zurich).

Berger, W. J.; and Saibel, Edward. Power series inversion of the Leontief matrix. Econometrica **25** (1957), 154–165.

A square matrix $L=I-A$ of order n is said to be Leontief when $a_{kk}=0$ for $k=1, 2, \cdots, n$, when $0\leq a_{ij}<1$ for $i\neq j$, and when $\sum_{i=1}^{n} a_{ij}<1$ for $j=1, 2, \cdots, n$. It is desired to invert L using power series. The expansion $L^{-1}=\sum_{k=0}^{\infty}A^k$ has been suggested by Waugh. The authors introduce $U=F_nD_n+dI$ and $V=D_nF_n+dI$, where F_n is a square matrix of order n having unit elements, D_n is a diagonal matrix having diagonal elements c_i, and c_i and d are defined so that $[N(L-U)]^2$ is a minimum. Then $N(L-U)<N(A)$ and $N(L-V)<N(A)$ so that the series expansion $L^{-1}=X^{-1}\sum_{k=0}^{\infty}(-EX^{-1})^k$ with $E=L-U$ or $L-V$ is preferred to $L^{-1}=\sum_{k=0}^{\infty}A^k$. A final scheme uses the W series for L^{-1}, where $W_1=\frac{1}{2}(U^{-1}+V^{-1})$ is the first approximant to L^{-1}. *P. S. Dwyer.*

★**Fischbach, Joseph W. Some applications of gradient methods.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis pp. 59–72. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

After a short description of the classical gradient method for solving nonlinear and linear systems of equations the author gives a short account of the method of conjugate gradients for linear systems. The relative merits of the two procedures are discussed based on some experiences with ORDVAC computations. In the second part of the paper both methods are applied for solving an ordinary differential equation

$$R(x, y, y')=\frac{dy}{dx}-f(x, y)=0. \tag{1}$$

The basic principle is to minimize the functional

$$J(y)=\int_0^1 R^2(x, y, y')dx \tag{2}$$

either by gradients or conjugate gradients replacing derivatives and integrals by an assumed difference scheme. It is an interesting idea of the author to use such a method of suvcessive approximation even in the case of an ordinary differential equation (1) of first order, where classical step integration is available. It would be interesting to learn if the minimizing of (2) is numerically more stable than the straightforward step integration of (1). *E. Stiefel* (Zurich).

Nagler, H. On the simultaneous numerical inversion of a matrix and all its leading submatrices. Math. Tables Aids Comput. **10** (1956), 225–226.

Let $A=X'DY$, where X and Y are upper triangular matrices whose diagonal elements are unity. Then

$$A^{-1}=Y^{-1}D^{-1}X^{-1}.$$

If A_r^{-1} denotes the first r rows and columns of A^{-1}, etc., it is observed that $A_r^{-1}=Y_r^{-1}D_r^{-1}X_r^{-1}$. The latter equation is interpreted as saying that any element of A_r^{-1} can be obtained by breaking off after r products an inner product defining A^{-1}.

G. E. Forsythe (Stanford, Calif.).

Pugačev, B. P. On two methods of approximate calculation of eigenvalues and eigenvectors. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 334–337. (Russian)

Let A be a positive-definite bounded self-adjoint operator on a real Hilbert space H, with λ_1 and λ_n $(0<\lambda_1<\lambda_n)$ as spectral bounds. For arbitrary x_0 in H, and $\alpha=1$ or 2, define a sequence $\{x_k\}$ inductively as follows: $\mu_k=(Ax_k, x_k)/(x_k, x_k)$, $\Delta_k=Ax_k-\mu_kx_k$, and

$$x_{k+1}=x_k-\alpha(\Delta_k, \Delta_k)(A\Delta_k, \Delta_k)^{-1}\Delta_k. \tag{*}$$

Geometrical interpretations are given and various results are announced: I. For $\alpha=1$ or 2, $\mu_0\geq\mu_1\geq\mu_2\geq\cdots$. II. For $\alpha=1$ or 2, if $\|E_{\lambda_1+\varepsilon}x_0\|>0$ for all $\varepsilon>0$, then $\|Ax_k-\lambda_1x_k\|\to 0$.

Henceforth assume λ_1 is an isolated point of the spectrum of A. Define $\sin^2(x; y)=1-(x/\|x\|, y/\|y\|)^2$. Let H_1 be the set of eigenvectors belonging to λ_1, and let the projection of x_0 on H_1 be $e\neq 0$.

Theorems: III. For $\alpha=1$ or 2, $\sin(x_k; e)\to 0$. V. For all $\varepsilon>0$, and $\alpha=2$, $\sin(x_{n+k}; e)\leq(q+\varepsilon)^k\sin(x_n; e)$, where $q<1$ and q is independent of ε. VI. Let A be a finite matrix with fixed λ_2, λ_n. Let x_0 be uniformly distributed in a certain sphere on R^n. Then, for λ_1 sufficiently small, $\alpha=2$, there is a positive probability that $\sin(\Delta_k; e_n)\to 0$.

Connections are shown with the "method of normal chords" [V. N. Kostarčuk, same Dokl. (N.S.) **98** (1954), 531–534; MR **16**, 863], and with the method of steepest descent [L. V. Kontorovich, Uspehi Mat. Nauk (N.S.) 3 (1948), no. 6(28), 89–185; MR **10**, 380]. Theorem VIII. If the x_k are computed by the method of steepest descent, $\sin(x_k; e)\to 0$. *G. E. Forsythe* (Stanford, Calif.).

Heinrich, Helmut. Bemerkungen zu den Verfahren von Hessenberg und Voetter. Z. Angew. Math. Mech. **36** (1956), 250–252.

This and the following paper are summaries of talks given at the Stuttgart meeting of the Gesellschaft für angewandte Mathematik und Mechanik, May 1956. The principal contribution is a certain systematization of Voetter's method [Z. Angew. Math. Phys. 3 (1952), 314–316; MR **14**, 501] of computing the characteristic polynomial of a matrix by desk computation. The procedure is described for a general matrix of order 4. Checks are indicated. There is also a statement comparing this with Hessenberg's method [R. Zurmühl, Praktische Mathematik..., Springer, Berlin, 1953; MR **15**, 407].

G. E. Forsythe (Stanford, Calif.).

Hirschleber, A. Ausnahmefälle des Graeffeschen Verfahrens. Z. Angew. Math. Mech. **36** (1956), 254–255.

In carrying out Graeffe's process for computing the zeros of a polynomial $P^{(0)}(z)$ with real coefficients one

needs to know the multiplicity of the moduli of the zeros. Let $P^{(n)}(z)=\sum_{r=0}^{p} A_r^{(n)}z^{p-r}$, where

$$A_p^{(n)}=(A_p^{(n-1)})^2+\sum_\nu (-1)^\nu 2A_{p-\nu}^{(n-1)}A_{p+\nu}^{(n+1)},$$
$$A_0=1,\ A_p\neq 0.$$

Order the zeros of $P^{(0)}$ so that $|z_1|\geq|z_2|\geq\cdots$. The author states that $|z_p|>|z_{p+1}|$ implies that

$$(*)\qquad \sum_\nu \lim_{n\to\infty} A_{p-\nu}^{(n)}A_{p+\nu}^{(n)}(A_p^{(n)})^{-2}=0.$$

An example shows that (*) is not sufficient for $|z_p|>|z_{p+1}|$, and this began the investigation reported here.

The author characterizes all exceptional cases in which (*) is not sufficient: namely, $P^{(0)}(z)$ or some Graeffe transform $P^{(n)}(z)$ is divisible by at least one cyclotomic polynomial of odd degree: $(z^j+\alpha)$ (j odd). Equivalent criteria for the exceptional cases are given. A later study is to consider questions of practical computation.

G. E. Forsythe (Stanford, Calif.).

Weinberger, H. F. Upper and lower bounds for eigenvalues by finite difference methods. Comm. Pure Appl. Math. 9 (1956), 613–623.

While the methods of the article will work for higher order operators like the biharmonie operator, attention is here restricted to the membrane problem $\Delta u+\lambda u=0$ in a two-dimensional region R, with $u=0$ on the boundary $\dot{R}$.

Let R_h be a slightly larger net domain containing R and all its left and downward translates of distance at most h. Let $\lambda^{(h)}$ solve the conventional finite difference problem $\Delta_h w+\lambda^{(h)}w=0$ on R_h, with $w=0$ on the boundary of R_h. Here

$$\Delta_h w=h^{-2}[w(x+h, y)+w(x-h, y)+w(x, y+h)+ w(x, y-h)-4w(x, y)].$$

The basic result is that $\lambda^{(h)}\leq\lambda$, and there is a beautiful short proof using a variational principle and Schwarz's inequality. {For some simple regions, $\lambda^{(h)}=\lambda-O(h)$ — Reviewer.}

The same technique shows that $\lambda^{(h)}\leq\lambda^{(h/2)}$. With careful estimation of neglected terms the author shows even more:

$$(*)\qquad (1+4^{-1}\rho^{-1}h)\lambda^{(h)}\leq\lambda^{(h/2)}[1-(32)^{-1}\lambda^{(h/2)}h^2],$$

where ρ is at most equal to the side of the smallest square containing $R_{h/2}$. Partial use of (*) can obtain or somewhat improve a related inequality of Hersch [C. R. Acad. Sci. Paris **240** (1955), 1602–1604; MR **16**, 929].

Using a related upper bound of Pólya [ibid. **235** (1952), 995–997; MR **14**, 656], the author now shows that the eigenvalue $\bar{\lambda}^{(h)}$ for a net region $\bar{R}_h\subset R$ can furnish the upper bound $\lambda\leq\bar{\lambda}^{(h)}[1-3h^2\bar{\lambda}^{(h)}]^{-1}$. Here $\bar{\lambda}^{(h)}$ is computed for $\bar{R}_h$ just as $\lambda^{(h)}$ was computed for R_h.

The author now shows in two ways that for special (e.g., convex) regions R it is possible to compute $\bar{\lambda}^{(h)}$ directly from $\lambda^{(h)}$, thus yielding arbitrarily close two-sided bounds for λ from one finite-difference calculation with a mesh constant h which can be estimated a priori.

G. E. Forsythe (Stanford, Calif.).

Forsythe, George E. Difference methods on a digital computer for Laplacian boundary value and eigenvalue problems. Comm. Pure Appl. Math. 9 (1956), 425–434.

Consider with the continuous eigenvalue problem (1) $\Delta u+\lambda u=0$ in a plane region R, where $u=0$ on the boundary C, Δ being the Laplacian, the discretized problem (2) $\Delta_h u+\lambda_h u=0$ on R_h, $u=0$ on C_h, where h is the side of the square meshes of a net **N**. Let $\lambda^{(i)}$ ($i=1, 2, \cdots$) denote the eigenvalues of (1) and $\lambda_h^{(i)}$ ($i=1, 2, \cdots, M(h)$) those of (2). When R is a rectangle bounded by lines of **N**, we have $\lambda_h^{(k)}<\lambda^{(k)}$. In the first part of this paper the author announces results of the form

$$\lambda_h^{(k)}\leq\lambda^{(k)}-\gamma^{(k)}h^2+o(h^2),$$

for certain regions R, where the $\gamma^{(k)}$ are certain constants which are positive if R is further restricted. That this result is in a certain sense best possible, follows from calculations in the rectangular case. The case $k=1$ has been considered by the author [Pacific J. Math. **4** (1954), 467–480; **5** (1955), 691–702; MR **16**, 179; **17**, 373], H. F. Weinberger [see the paper reviewed above], J. Hersch [C. R. Acad. Sci. Paris **240** (1955), 1602–1604; MR **16**, 929] and V. K. Saul'ev [Vyčisl. Mat. Vyčisl. Tehn. **2** (1955), 116–166; MR **16**, 1056].

In the second part, there is a report on actual experience in solving on SWAC the problem (3) $\Delta_h=0$, u given on C_h, as well as the problem (2). The process of successive overrelaxation has been found effective. For instance, when R_h is a rectangle with 30×68 interior points, one sweep takes 8.5 seconds. With pure relaxation, about 2300 sweeps would be needed to reduce the error by a factor of 10^{-6}; with optimal relaxation only about 90 sweeps would be required. The codes available are quite elaborate; they can handle curvilinear boundaries and various boundary conditions.

J. Todd.

Kislicyn, S. G. On finding of a periodic solution of the equation $y'=f(x, y)$ by Newton's method. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 5(9), 101–104. (Russian)

Let $f(x, y)$ be continuous, with $f(x+1, y)=f(x, y)$. Assume $-M\leq\partial f/\partial y\leq -m$ and $|\partial^2 f/\partial y^2|\leq H$. A Newton's method is proposed for finding a periodic solution y of $y'=f(x, y)$. Given $y_1(x)$, periodic, determine a periodic solution y_2 of the perturbed equation

$$y_2'=f(x, y)+(y_2-y_1)\partial f/\partial y.$$

Get $y_3, y_4, \cdots$ similarly.

If $|y_1-y|$ is small enough, it is proved that $|y_{n+1}-y_n|$, and hence also $|y_n-y|,\to 0$ quadratically. Pointwise bounds for $|y-y_n|$ are given in terms of m, M, H, etc. There is a numerical example. No discussion of possible computational difficulties is given.

G. E. Forsythe.

★ **Bennett, Albert A.; Milne, William E.; and Bateman, Harry. Numerical integration of differential equations.** Dover Publications, Inc., New York, N.Y., 1956. 108 pp. $1.35.

This is an unabridged and unaltered photographic reprint of a report by the same authors [Bull. Nat. Res. Council **92** (1933)].

Chapter 1 (by Bennett) contains numerous formulas for polynomial interpolation of functions of one real variable, without remainder formulas {!}, and an introduction to finite-difference methods. Chapter 2 (by Bennett) is a historical and bibliographical survey of works dealing with successive approximations for solving equations and differential equations. Chapter 3 (by Milne) is the only chapter really coming to grips with the subject of the title. Methods named for Adams, Moulton, Steffensen, Milne, and Runge-Kutta are described, with

accompanying numerical examples. Chapter 4 (by Bateman) gives a historical survey of papers written on the transition from difference-equation solutions to differential-equation solutions, on Ritz's method, and on expansions of solutions of ordinary differential equations in a parameter.

The 280-odd footnotes and the 250-odd general bibliographic entries indicate the importance of the book as a source of references. But the reader interested in how to integrate differential equations numerically on a desk computer will find little except in Chapter 3, and that material is covered by the same author in Milne's "Numerical solution of differential equations" [Wiley, New York, 1953; MR **16**, 864]. As for solving differential equations on an automatic digital computer, the first book has yet to be published. *G. E. Forsythe.*

Halilov, Z. I. Solution of filtration problem by the method of nets. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **12** (1956), 245–248. (Russian. Azerbaijani summary)

A propagation finite-difference scheme for solving a class of simultaneous non-linear partial differential equations arising in hydrodynamical problems is given. *C. Saltzer* (Syracuse, N.Y.).

Philip, J. R. Numerical solution of equations of the diffusion type with diffusivity concentration-dependent. II. Austral. J. Phys. **10** (1957), 29–42.

This paper deals with the equation $\theta_t=(D\theta_x)_x-K_x$ $(x>0, t>0)$ with $\theta(0, t)=\theta_0$ and $\theta(x, 0)=\theta_n$, where D and K are given functions of θ, and θ_0 and θ_n are given constants. Extending a method used previously by the author [Trans. Faraday Soc. **51** (1955), 885–892; MR **17**, 196], the treatment consists in first transforming the problem to one in which θ and t are taken as the independent variables. The governing equation becomes $x_t+(D/x_\theta)_\theta=K_\theta$, while the auxiliary conditions are to be considered as expressed in the forms $x(t, \theta_0)=0$ and $\int_{\theta_n}^{\theta_0} x d\theta=-\int_0^t [D(\theta_0)/x_\theta(t, \theta_0)]dt+[K(\theta_0)-K(\theta_n)]t$. (The last equations is not given explicitly by the author.) The subsequent analysis is equivalent to assuming a solution in the form $x=\sum_1^\infty f_m(\theta)t^{m/2}$, obtaining a set of ordinary differential equations and associated conditions governing the coefficient functions by substitution and matching of coefficients of successive powers of $t^{1/2}$, and solving these equations recursively by numerical finite-difference methods. Whereas the general questions of convergence and of approximation-error magnitudes are not considered in detail, a specific example (D=const, K_θ=const) is included in which the known exact solution and a three-term approximate solution are compared graphically for two values of t, with apparently satisfactory agreement. Applications of the technique to physical problems are to be published elsewhere. *F. B. Hildebrand.*

van Wijngaarden, A. Session on numerical analysis. I. Proc. Inst. Elec. Engrs. B. **103** (1956), supplement no. 1, 112–113, discussion 120–122.

A class of partial difference equations on a grid which approximate linear elliptic partial differential equations is considered. An iteration procedure is described together with a relaxation method. The convergence of the iteration procedure is proved and bounds are given for the error. An application to the Laplace partial difference equation is given as an example. The discussion contains a method for obtaining quadratic convergence and an application to the Poisson partial difference equation. *C. Saltzer* (Syracuse, N.Y.).

Lyusternik, L. A. On the difference analogue of Green's function for the Laplace operator. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. **1** (1956), 43–53. (Russian)

Consider a finite difference net of points (mh, m_1h) in an oblique coordinate system (ξ, η) in the plane. Let Δ_h be any finite difference approximation to the Laplacian Δ over the net, such that $\Delta_h u(\xi, \eta)=\Delta u(\xi, \eta)+o(h^2)$, as $h\to 0$. For $0\leq m, m_1<2n$ the grid points of the net define a parallelogram R. The author defines and gives an explicit representation for a finite-difference Green's function $g_h(A, B)$ over R. He proves that, as $h\to 0$,

$$g_h(A, B)=-2\pi^{-1}\ln(\overline{AB})+O(1).$$

For any region Q inside R, a finite difference Green's function $g_h(A, B)_Q$ is defined. For $\rho<0$, let S_ρ be a disk of radius ρ centered at A. It is proved that for $\rho>0$, as $h\to 0$, $g_h(A, B)_Q\to g(A, B)_Q$ uniformly for B in $Q-S_\rho$, where $g(A, B)_Q$ is the ordinary Green's function for Q.

The basic method is the expansion of g_h and g in functions $\exp[\pi i(x\xi+y\eta)]$, which, as functions of ξ, η, are eigenfunctions both of Δ and Δ_h. *G. E. Forsythe.*

Davidenko, D. F. On a difference method of solution of the Laplace equation with axial symmetry. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 910–913. (Russian)

The basic computing problem is the use of finite difference methods to solve the Dirichlet problem for Laplace's equation $\Delta u=r^{-1}u_r+u_{rr}+u_{zz}=0$ in spatial problems with axial symmetry. The author's contribution here is a systematic method for producing difference approximations to Δu for irregular nets. The approach is to expand $u(r, z)$ locally in an infinite series of harmonic functions of complicated form. The error term is given, but not appraised, for an irregular net.

The special case of a square net with mesh constant h is treated explicitly by means of Legendre polynomials. A nine-point formula is given, The coefficients are certain polynomial functions of h/r (for $r>0$), and constants, for $r=0$. The error term $\Delta(u)-\Delta_h(u)$ is stated to be $O(h^8)$ for $r\geq h$, and $O(h^6)$ for $r=0$.

A paragraph summarizes the greater precision obtained by these formulas for a simple problem, compared with the use of simpler formulas of Geršgorin [Ž. Prikl. Fiz. **6** (1929), no. 3–4, 3–30]. *G. E. Forsythe.*

Douglas, Jim, Jr.; and Rachford, H. H., Jr. On the numerical solution of heat conduction problems in two and three space variables. Trans. Amer. Math. Soc. **82** (1956), 421–439.

The transient heat flow equation is approximated on a grid by a finite difference method which leads to simple systems of implicit equations for successive time instants. An eigenvalue analysis of the discrete equations shows that the process is stable for arbitrary space- and time-increments, and the convergence of the solution of the difference equation to the solution of the partial differential equation is deduced from the stability of the approximating system. This system of equations is also applied to the steady-state problem; i.e., the Dirichlet problem for the Poisson equation by introducing a sequence of parameters to replace the time increment. The convergence of the resulting iterative process is

proved for a general class of sequences by eigenvalue analysis and the choice of the parameters for optimizing the rate of convergence is discussed. *C. Saltzer.*

Cyrlin, L. È. Some questions of the mathematical theory of coronal discharge in the case of constant voltage. Ž. Tehn. Fiz. **26** (1956), 2524–2538. (Russian)

The physical problem under consideration requires the solution of a non-linear partial differential equation of the third order. An approximate analytical solution is found by an adaption of the "method of subdomains" [see L. Collatz, Numerische Behandlung von Differentialgleichungen, 2nd ed., Springer, Berlin, 1955, p. 29; MR **16**, 962]. The solution is compared with experimental results. No attempt is made to estimate the error. *P. Henrici.*

★**Zagustin, A. Ecuaciones integrales y el calculo numerico de vigas. [Integral equations and numerical calculation of beams.]** Universidad Central de Venezuela, Facultad de Ingenieria, Caracas. 21 pp.

Zurmühl, Rudolf. Behandlung der Plattenaufgabe nach dem verbesserten Differenzenverfahren. Z. Angew. Math. Mech. **37** (1957), 1–16. (English, French and Russian summaries)

Assuming the plate to be rectangularly bounded, the plate problem $\Delta\Delta u=p/K$ is treated by means of an improved calculus of finite differences which permits both the treatment of the boundary conditions and the determination of the 2nd derivatives, i.e., the bending moments, with high accuracy. The application of the formulae to the various possible ways of supporting the plate is discussed and illustrated by numerical examples. A comparison with known exact solutions demonstrates the accuracy of the method.

Author's summary.

See also: Hunter, p. 791; Lavrent'ev, p. 799; Wynn, p. 801; Schaefer, p. 811; Pearcey, p. 829; Peltier, p. 831; Landau, p. 831; Kron, p. 831; Tabaks, p. 831; Friedmann, p. 851.

Graphical Methods, Nomography

Lipatova, D. L.; and Džems-Levi, G. E. Standardization of projective transformations. Moskov. Gos. Univ. Uč. Zap. **181**. Mat. 8 (1956), 235–240. (Russian)

Projective transformations of nomograms are often necessary. The author suggests a computational scheme for the determination of the coefficients of a transformation in which two quadrangles correspond to each other. *E. Lukacs* (Washington, D.C.).

★**Konorski, Bolesław; i Krysicki, Wlodzimierz. Nomografia. [Nomography.]** Państwowe Wydawnictwa Techniczne, Warszawa, 1956. 367 pp. zł. 24.20.

Weiss, L. L. A nomogram for log-normal frequency analysis. Trans. Amer. Geophys. Union **38** (1957), 33–37.

See also: James-Levy, p. 834; Herovanu, p. 857.

Tables

★**Лебедев, А.В. и Федорова, Р.М. [Lebedev, A. V.; and Fedorova, R. M.] Справочник по математическим таблицам. [Reference book on mathematical tables.]** Izdat. Akad. Nauk SSSR, Moscow, 1956. xlvi+552 pp. 29.20 rubles.

One of the most valuable books in the computer's library is an index of mathematical tables. The first such modern volume was the scholarly and comprehensive Fletcher, Miller, and Rosenhead, "An index of mathematical tables" [McGraw-Hill, New York, 1946; MR **8**, 286], here referred to as FMR. The book has been out of print for several years, and computers have been anxiously awaiting a second edition. The next to appear was K. Schütte's "Index of mathematical tables..." [Oldenbourg, München, 1955; MR **17**, 414], a short and highly selective list.

The volume by Lebedev and Fedorova, here denoted by LF, is comprehensive, and in this respect is essentially a Russian version of FMR, brought up to date to about 1954. Like FMR, but unlike Schütte, it is limited to mathematical functions and omits empirical functions. Like FMR, LF has two major parts: I. an outline of functions, in which detailed characteristics of the tables are presented (number of digits in the functional values, range and interval of the argument), together with a reference to the tables; II. a bibliographic section with detailed citations of the tables.

In part I there are two items in FMR which are missing in LF. First, the scholarly notes about errors, existence of differences, etc. The other is the author's name and date, together with bold face type for the standard tables. In LF there are only bracketed numbers referring to part. II. LF indexes a good many tables of products mn (for m, n integers) and other very elementary material which FMR treats rather lightly.

In LF part II is grouped in 14 different sections, paralleling the outline of part I. LF furnishes titles to journal articles, which FMR does not. LF also indicates which of the principal Leningrad or Moscow libraries have the title — a valuable feature in a bibliography. Because the entries are not arranged in a single alphabetical order, LF furnishes an index to part II by author's names. FMR has no index to part II, but follows a single alphabetical order.

A major difference in using the two books is that LF has no alphabetical index to functions to part I, while FMR does. To fill this need LF has a 41-page table of contents to part I, with names and formulas of the tables listed. This puts some burden on the reader to understand the general classification of functions. The notation generally seems to be consistent with FMR.

As sources, LF mentions Referativnyĭ Žurnal (Matematika), Math. Tables Aids Comput., FMR, and H. T. Davis and V. Fisher, A bibliography of mathematical tables [Evanston, Illinois, 1949]. Spot checking suggests that LF has used almost all the entries in FMR. (On p. 506 they repeat an FMR error of referring to K. Hidaka as F. Hidaka.)

The reviewer noticed a number of pre-1940 Soviet tables not listed in FMR, but they seem usually to be logarithms or even more elementary. However, the most important new entries are the very numerous post-1945 tables. Both in the Soviet Union and in the West the new computing machines are turning out tables at an accelerating rate.

Typography and paper seem very good. Casual inspection reveals a number of minor misprints, in decided contrast to the meticulous FMR. On page 540, for example, the last entry for Bessel should be 'XIV, 182', while 'Bennet' should read 'Bennett'. Cross-references from part II to part I would increase the value of LF.

In summary, this book is a must for every working computer's library, and will surely remain so at least until FMR produce a new edition. The prospective purchaser need not be deterred by not reading Russian, as the great majority of the material actually used is in universal mathematical symbolism. Western references are in the Roman alphabet.

In spite of the announced 8000-copy edition, experience suggests that the book will be very difficult to find after a couple of years. *G. E. Forsythe.*

Horgan, R. B. Table of coefficients for the partial summation of series. Math. Tables Aids Comput. **10** (1956), 156–162.

The coefficients $A_m(n)$ of H. E. Salzer's paper [same journal **10** (1956), 149–156; MR **18**, 5] are tabulated here to fifteen decimals, of which the last figure is expected to have a maximum error of one unit. This might be expected to facilitate the computation of S_n, but the table is not arranged in the most convenient way for this purpose. *L. Fox* (Berkeley, Calif.).

★**Fröberg, Carl-Erik. Hexadecimal conversion tables.** Lund University, Department of Numerical Analysis, Table No. 1. CWK Gleerup, Lund, 1957. 26 pp. 3 kr.

This is a completely revised version of the booklet of the same title issued in 1952 [MR **14**, 410]. The first part of the table gives the hexadecimal (i.e., scale 16) representation of the integers $n=1(1)1024(16)4096$ and $n=10^2(10^2)10^3(10^3)10^4\cdots10^{12}(10^{12})10^{13}$. The second part gives hexadecimal equivalents of the decimal numbers $10^{-18}(10^{-18})10^{-16}(10^{-16})10^{-14}\cdots10^{-2}(10^{-2})1$; the errors do not exceed 4 units in the last place. Here a hexadecimal number consist of 60 binary digits assembled as 15 hexadecimal digits $(0, 1, \cdots, 9, A, B, \cdots, F)$, the first binary digit representing the sign, and the binary point following this binary digit. The third part is a conversion table for decimals to floating hexadecimal numbers of the form $z=x\cdot 2^y$, where $\frac{1}{2}\leq x<1$ and where y is an integer. Actually x is given in the standard form just described and $\bar{y}=128+y$ is given as a 2 hexadecimal digit number; this means that a range of about 10^{-38} to 10^{+38} is covered. There are given $z=n\cdot 10^{\pm k}$ for $k=0(1)12$, $n=1, 3, 7, 9$ and for $k=13(1)25$, $n=1$. In addition to the standard representation of z, three other representations, $z=(x/2^i)\cdot 2^{y+i}$ $(i=1, 2, 3)$, are given; these are often convenient in manual operations. The fourth part is a page of constants given in straight and floating hexadecimal notation. The fifth part is a conversion table from hexadecimal to 20 digit decimal numbers, for $z=n\cdot 16^{-k}$ when $n=0(1)F$, $k=1(1)15$.

The second, third and fifth tables have been computed on SMIL, the computer of the University of Lund. The tables are all clearly printed. *John Todd.*

★**Hof, Hans. Powers, roots, reciprocals, from .0001—15000. 1st ed.** Professional Supply Company, P. O. Box 162, Jenkintown, Pa., 1956–1957. iii+200+iii+300+i+24 pp.

The first part gives, for $n=.0001(.0001)1$, the exact values of n^2, n^3, n^4 and $n^{1/2}$ to 10D, $n^{1/3}$ to 9D, $n^{1/4}$ to 8D up to $n=.2$, thereafter to 7D, n^{-1} to 7D up to $n=.1$, thereafter to 8D. The second part gives, for $n=1(1)15000$, exact values of n^2, n^3, n^4 and $n^{1/2}$ to 8D, $n^{1/3}$ to 6D, $n^{1/4}$ to 6D, n^{-1} to 10D. In addition, for $n=15010(10)101000$, there is $n^{1/2}$ to 7D and $n^{1/4}$ to 8D. In both these parts a range of 50 values of n is covered on a page.

Finally, there are the exact values of n^r for $r=5(1)10$, $n=1(1)100$ and for $n=11(1)20$, $n=1(1)10$.

The printing is clear, especially in the first part, which is printed in red. It is stated that the roots are correct to within a unit in the last place. There is no description of the construction or checking of the tables. *John Todd.*

★**Ōishi, Sanshirō. Numerical intersection charts for calculations.** (Japanese and English). The Universal Management Administration Institution, Tokyo, 1957. ii+87 pp.

These charts are intended for numerical calculations by nonmathematical users. They consist of (i) a multiplication table giving the product ab for $a=100(1)1000$, $b=1(1)100$; (ii) a table giving n^2 for $n=0(1)999$, (iii) a table giving $\sqrt{n}$ for the same range. The first of these has the products, in slant array to facilitate its use in giving squares, cubes, roots and reciprocals.

J. Riordan (New York, N.Y.).

★**Pearcey, T. Table of the Fresnel integral to six decimal places.** Cambridge University Press, New York, 1956. 63 pp. $2.50.

Let $C(u)+iS(u)=\int_0^x \exp(1/2i\pi t^2)dt$; let $u=(2x/\pi)^{\frac{1}{2}}$. Then C and S are tabulated as functions of x as follows: $x=0(.01)1$, 7D, $x=1(.01)50$, 6D. Second central differences are given, except for the first few entries. The tables were mainly constructed by sub-tabulation to one-fifth and then to one-tenth (using a National Type 3000 accounting machine) of the well-known tables of Lommel [cf. G. N. Watson, Theory of Bessel functions, Cambridge, 1944, p. 744; MR **6**, 64] which were checked by differencing; these tables give 7D values for $x=0.02(.02)1$ and 6D for $x=0.5(.5)50$. Values for small x were obtained using the series expansion. It is stated that errors rarely amount to a unit in the last place. A table giving corrections to allow for the contribution of second differences is included. The preface mentions (through references to Math. Tables Aids Comput.) various other tables; a more recent table is that of Akad. Nauk SSSR [Tables of Fresnel integrals, Moscow, 1953; MR **16**, 523].

The present tabulation was made to assist in diffraction calculations by permitting linear interpolation over a wide range. *John Todd* (Washington, D.C.).

Nielsen, Jack N. Tables of characteristic functions for solving boundary-value problems of the wave equation with application to supersonic interference. NACA Tech. Note no. 3873 (1957), 245 pp.

"Tables are presented containing 69,000 values of a set of characteristic functions of two variables which first arose in problems of supersonic wing-body interference. The functions solve boundary-value problems of the second kind for the wave equation in three dimensions with circular cylindrical boundaries or problems of the unsteady heat-conduction equation in two space dimensions with circular boundaries. The functions themselves have the physical significance of cylindrical pressure waves. The tables have extensive use in problems of aerodynamic interference at supersonic speeds."

The functions which are tabulated are

$$W_m(x, r)=L^{-1}[r^{-\frac{1}{2}}+e^{s(r-1)}K_m(sr)/K_m'(s)],$$

where L^{-1} denotes the inverse Laplace transform. The tabulations were carried out by W. A. Mersman and S. Crandall and are over the following ranges: $m=0, 1$, $x=0(.01)10$; $m=2, 3$, $x=0(.01)7$; $m=4(1)10$, $x=0(.01)5$, all for $r=1, 1.1, 1.25, 1.5, 2, 3, 4, 6, 8, 10$. For $0\leq x\leq 1.2$, 8D are given; for $1.2\leq x\leq 1.5$, 6D are given and for $x>1.5$, 4D are given. It is stated that the results are good to at least 4D except in certain small areas where an error of up to .001 is possible. The tables were calculated from power series expansions in x for $0\leq x\leq 1.5$. For $x\geq 1.5$ and $m=0(1)5$, W_m was computed as the solution of an integral equation. For $x\geq 1.5$ and $m>5$, a method of characteristics was used.

Several examples illustrating the use of the tables in aerodynamic calculations are worked out in some detail. A detailed report on the method of calculations is to appear; after pilot studies on an IBM CPC, the main calculations were carried out on an IBM 701 and a Datatron 204. *John Todd* (Washington, D.C.).

Sherman, B. Percentiles of the ω_n statistic. Ann. Math. Statist. 28 (1957), 259–261.

Let n points be selected independently from a uniform distribution on a unit interval and let L_k $(k=1, 2, \cdots, n+1)$ be the length of the kth subinterval determined. Let

$$\omega_n=\frac{1}{2}\sum_{k=1}^{n+1}\left|L_k-\frac{1}{1+n}\right|.$$

The author has previously determined the distribution function of ω_n and shown that it tends to normality as $n\to\infty$ [same Ann. 21 (1950), 339–361; MR 12, 192]. He now presents tables, computed on SWAC, for the 90, 95 and 99 percentiles of this and certain related normalized distributions for $n\leq 20$. The tables show that the approach to normality is rather slow. The definition of $\omega_n^{(1)}$ on p. 261 should read $\omega_n^{(1)}=(\omega_n-E_n)/D_n$. *L. Moser.*

See also: Interpolation and allied tables, p. 824.

Machines and Modelling

Penndorf, Rudolf B. New tables of Mie scattering functions for spherical particles. VI. Total Mie scattering coefficients for real refractive indices. Air Force Cambridge Research Center, Bedford, Mass., Geophysical Research Papers No. 45 (1956), ix+98 pp.

This volume is based on extensive computations (made using an IBM 701) of the total Mie scattering coefficient. This coefficient K has been computed for $\alpha=0(.1)30$ for indices of refraction $n=1.33, 1.40, 1.44, 1.686, 1.50$. The results are considered to be correct to 5S. Supplementary desk calculations (to 4S) have been made for $n=1+\varepsilon$, 2, ∞. There is a discussion of the theory and a critical bibliography. In addition, an empirical method for the calculation of K to within $\pm 3\%$ for arbitrary α, n is described. *John Todd* (Washington, D.C.).

Hinshelwood, Cyril. Opening address: Convention on digital-computer techniques. Proc. Inst. Elec. Engrs. B. 103 (1956), supplement no. 1, 1–2.

Opening address by the President of the Royal Society to the convention on digital computers held in London, April 9–14, 1956, by the Institution of Electrical Engineers.

Sir Cyril reviews the diverse streams of thought which have contributed to the subject of the convention, such as the development of formal logic and of theory of numbers, the discovery of the thermionic valve and of a host of electric and magnetic phenomena, and stresses that the philosophy of the conceptions was present in the originators of calculating machines almost from the beginning: mathematics was here not merely a tool but an essential ingredient of the problem. He touches on the scientific significance of computers in the solution of complex scientific problems and on their social significance in the mechanization of many human activities.

Finally, the encouragement and stimulation to various kinds of work which stems from the potentialities of computers is discussed by means of some specific examples, and the lecture closes with remarks on the analogy with human brains. *W. F. Freiberger.*

Williams, F. C. Convention on digital-computer techniques: introductory lecture. Proc. Inst. Elec. Engrs. B. 103 (1956), supplement no. 1, 3–9.

Professor Williams, in this introductory lecture to the convention on digital computers (see preceding review) discusses the history of the Manchester University Computing Machine Laboratory, and parallels it with developments at Cambridge and the National Physical Laboratory in England, in the United States and on the Continent.

The history extends from 1946 to the time of the convention and is a review of the applications developed over these years as well as of the machines themselves. A payroll application is worked out in considerable detail and significant general points arising in this program are given emphasis.

There follows a glimpse into the future; the next ten years are, according to the author, not likely to see advances in basic principles comparable to the past ten years, and improvements are expected to be mainly in design and technique. Also, he sees a retreat from universality and the emergence of machines directed at particular markets, and presents an interesting discussion of possible future applications — the field in which much progress can be expected. *W. F. Freiberger.*

Bullard, Edward. Convention on digital-computer techniques: introduction to the session on engineering and scientific applications of digital computers. Proc. Inst. Elec. Engrs. B. 103 (1956), supplement no. 1, 10–11.

The former director of the National Physical Laboratory contrasts, at this convention (see preceding two reviews), scientific activities in which the computer has a part with those in which it has no part to play, and discusses the intermediate field. In problems of the continuum, the relation between the net-size of finite difference methods and the number of arithmetic operations necessary is pointed out. Sir Edward is not certain that we shall ever be able to solve three-dimensional partial differential equations adequately by finite difference methods on computers and suggests a way out: not to solve the problem directly, but introduce the time as an additional variable, starting with some solution, and to integrate successively in time — letting the system settle, as it would in nature. *W. F. Freiberger.*

Peltier, Jean. Mécanisation des problèmes linéaires sur machines électroniques. C. R. Acad. Sci. Paris **244** (1957), 1003–1005.

A brief description of some programs for solving simultaneous linear algebraic equations and related problems by the pivodal method on a type 650 computer.
R. W. Hamming (Murray Hill, N.J.).

Lavine, L. R.; and Rollett, J. S. Crystal structure refinement by least-squares with the Electro-Data computer. Acta Cryst. **9** (1956), 269–273.

Two programs have been coded and used for least-squares refinement by the Electro-Data digital computer with three-dimensional data for crystal structures of classes $2/m$ and 222. Relevant details of the computer are given. The method of calculation and the form of input and output are described. The rate of operation of each program is stated. An expression for predicting the value of the minimisation function has been tested.
W. Nowacki (Bern).

Landau, H. G. A simple procedure for improved accuracy in the resistor-network solution of Laplace's and Poisson's equation. J. Appl. Mech. **24** (1957), 93–97.

By adding cross-connections between the node-points of a resistance network the truncation errors are reduced from an order of $O(h^2)$ to $O(h^6)$.
G. Kron.

Kron, Gabriel. How to use the A. C. network analyzer for "tearing". Matrix Tensor Quart. **6** (1956), 131–134 (1 plate).

A physical interpretation of the "Diakoptic" method leads to an adaption of Kron's method for use on analogue computers.
C. Saltzer (Syracuse, N.Y.).

Tabaks, K. K. Modelling in an electrolytic bath of some special cases of the Poisson equation. Latvijas PSR Zinātņu Akad. Vēstis 1956, no. 4(105), 145–148. (Russian. Latvian summary).

An analogue method based on the use of an electrolytic tank, conducting paper or iron plate is given for solving equations of the form: Laplacian of unknown function proportional to gradient of given function, where the given function itself satisfies Laplace's equation, or in particular is linear.
C. Saltzer.

See also: Hall, p. 816; Fischbach, p. 825; Forsythe, p. 826; Nielsen, p. 829; Sherman, p. 830.

PROBABILITY

Laga, [Laha], R. G. Characterization of a normal distribution in terms of properties of related linear statistics. Vestnik Leningrad. Univ. **11** (1956), no. 13, 90–98. (Russian)

This is a Russian version of the author's paper [Ann. Math. Stat. **28** (1957), 126–139; MR **18**, 768].

E. Lukacs (Washington, D.C.).

Takacs, Lajos. On the sequence of events, selected by a counter from a recurrent process of events. Teor. Veroyatnost. i Primenen. **1** (1956), 90–102. (Russian summary)

A sequence of events E_n occurring at time t_n is called a recurrent sequence if the time intervals t_n-t_{n-1} are independently distributed each with the distribution function $F(x)$. The author assumes that the first and second moment of $F(x)$ exist. The events E_n are recorded by a counter in the following way. (1) If at time $t_0=t_0'$ an impulse starts which has the duration χ_0, the event E_n starts an impulse with probability p if at time t_n an impulse is in progress and with probability 1 if no impulse is in progress. (2) If χ_n is the duration of the nth impulse the random variables χ_n are independently distributed each with the same distribution function $H(x)$. The impulses E_n' occurring at time t_n' also form a recurrent sequence. Let $G(x)$ be the distribution of t_n-t_n', ν_t the number of events recorded during the time interval $[0, t]$ and $P(t)$ the probability that at time t no impulse is in progress.

Under the assumption that χ_n is a constant the author determines the characteristic function of $G(x)$, the distribution functions of ν_t, $\nu_{T+t}-\nu_T$, and the limit distribution of $\nu_{\tau+t}-\nu_\tau$ in the case that $\tau=T$ and in the case that τ is uniformly distributed in $[0, T]$ when $T\to\infty$. The probability $P(t)$ is obtained and existence of $\lim_{t\to\infty} P(t)$ is discussed. The cases $p=0$ and $p=1$ are also discussed in the case that χ_n is not a constant.
H. B. Mann.

Čentsov, N. La convergence faible des processus stochastiques à trajectoires sans discontinuités de seconde espèce et l'approche dite "heuristique" au tests du type de Kolmogorov-Smirnov. Teor. Veroyatnost. i Primenen. **1** (1956), 155–161. (Russian. French summary)

The author proves the following two theorems: Theorem 1. Given a separable stochastic process $\xi(t)$ $(0\leqq t\leqq 1)$ satisfying the condition

$$(*)\qquad M|\xi(t_1)-\xi(t_2)|^p|\xi(t_2)-\xi(t_3)|^q<C|t_1-t_3|^{1+r},$$

where $p\geqq 0$, $q\geqq 0$, $r>0$, $t_1<t_2<t_3$, and C is a constant independent of t, then with probability one the trajectories of the process do not have any discontinuities of the second kind. Theorem 2. Let $\xi_n(t)$ be a sequence of stochastic processes which converge weakly to a separable stochastic process $\xi_0(t)$. Let $g(t)<f(t)$ be semi-continuous functions. Assume that (a) condition (*) is satisfied for every $\xi_n(t)$, the constant C being independent of n and t, and (b) as $z\to+0$

$$P\{g(t)+z\leqq\xi_0(t)\leqq f(t)-z,\text{ all } t\}\to P\{g(t)\leqq\xi_0(t)\leqq f(t),\text{ all } t\}.$$

Then as $n\to\infty$

$$P\{g(t)\leqq\xi_n(t)\leqq f(t),\text{ all } t\}\to P\{g(t)\leqq\xi_0(t)\leqq f(t),\text{ all } t\}.$$

These theorems provide a foundation for the heuristic approach to the Kolmogorov-Smirnov tests given by Doob [Ann. Math. Statist. **20** (1949), 393–403; MR **11**, 43]. This was also treated by Donsker [ibid. **23** (1952), 277–281; MR **13**, 853].
J. L. Snell (Hanover, N.H.).

Moustafa, M. D. Input-output Markov processes. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 112–118.

Two input-output models are considered in each of which the input distribution is independent of the number of objects in the reservoir. According as input follows or precedes output the transition matrix is the product of the input and output matrices in one order or the other. Ergodicity is proved under certain conditions.
K. L. Chung (Syracuse, N.Y.).

Darling, D. A.; and Kac, M. On occupation times for Markoff processes. Trans. Amer. Math. Soc. **84** (1957), 444–458.

Let $\{x_t, t \geq 0\}$ be an abstract-valued Markov process with the stationary transition probability $p(x, E; t)$. Let $p_s(x, E) = \int_0^\infty e^{-st} P(x, E; t) dt$; $V(x)$ be measurable and ≥ 0. Assumption: There exists $h(s) \to \infty$ as $s \to 0$ and $C > 0$ such that

$$(h(s))^{-1} \int V(y) p_s(x, dy) \to C \quad (s \to 0)$$

uniformly in $x \in \{\xi | V(\xi) > 0\}$. Then if $h(s) = L(s^{-1}) s^{-\alpha}$ $(0 \leq \alpha < 1)$ and $L(s^{-1})$ is slowly varying as $s \to 0$,

$$(Ch(t^{-1}))^{-1} \int_0^t V(x(\tau)) d\tau$$

has as limit the Mittag-Leffler distribution of index α. Conversely, if the above has a nondegenerate limit distribution after $Ch(t^{-1})$ is replaced by some $u(t) > 0$, then $u(t)$ is indeed $Ch(t^{-1})$, where h is given in the assumption and must be of the stated form. There is a discrete analogue. The connexion with Karamata's Tauberian theorem is discussed. Applications to sums of independent random variables unify some known results but under more restrictive conditions than otherwise obtainable.

K. L. Chung (Syracuse, N.Y.).

Sarmanov, O. V. Necessary and sufficient conditions of existence of a discrete limit law for Markov chains with two states. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 735–738. (Russian)

Consider an infinite sequence of not temporally homogeneous Markov chains with two states A and $\bar{A}$. Let $P_{s,n}$ denote the probability that A occurs s times in the first n steps of the nth chain. Conditions are found so that $\lim_{n \to \infty} P_{s,n} = P_s$, $s \geq 0$; and $\sum_{s=0}^\infty P_s = 1$. *K. L. Chung.*

Smith, Walter L. On renewal theory, counter problems, and quasi-Poisson processes. Proc. Cambridge Philos. Soc. **53** (1957), 175–193.

It is shown that the standard machinery of renewal theory leads directly to some results by Hammersley [same Proc. **49** (1953), 623–637; MR **15**, 139]. A quasi-Poisson process is one in which the residual life time at y has a distribution not depending on y for $y \geq$ some fixed value. Two methods of censoring a renewal process to yield a new renewal process are considered. *K. L. Chung.*

Santoboni, Luigi. Le assicurazioni di annualità su una o due teste, con riferimento all'assicurazione mista e al termine fisso. Archimede **8** (1956), 263–271; **9** (1957), 20–25.

See also: Monin, p. 836; Gel'fand, p. 859; Syski, p. 859; Pinsker, p. 859; Kemeny, p. 860.

STATISTICS

Agnew, Ralph Palmer. Estimates for global central limit theorems. Ann. Math. Statist. **28** (1957), 26–42.

Given n independent random variables, $\{\xi_k\}$ with the same distribution function (d.f.)

$$F(x) = \Pr\{\zeta_k \leq x\},$$

with mean 0 and variance 1. The author considers

$$C_n^{(p)} = \int_{-\infty}^\infty |F_n(x) - \phi(x)|^p dx,$$

where $F_n(x)$ is the d.f. for $\sum_{k=1}^n \zeta_k / \sqrt{n}$ and $\phi(x)$ the normal d.f.

A general formula is given for $p=2$ in terms of the characteristic function of $F(x)$. Special formulas are derived for $p=2$: for $F(x)$ a symmetric binomial d.f., C is $O(1/n)$; for $F(x)$ a uniform d.f., C is $O(1/n^2)$.

R. L. Anderson (Raleigh, N.C.).

Uzawa, Hirofumi. Note on the rational selection of decision functions. Econometrica **25** (1957), 166–174.

This paper constitutes a generalization of the main result of a paper of the reviewer's [Econometrica **22** (1954), 422–443; MR **16**, 271]. In that paper it was shown that in the case of a finite number of states of nature certain natural axioms of "rationality" lead to the Laplace criterion for selecting among decision functions. The author extends this result to the case where there are a countable number of states of nature and to the case where the states of nature are the elements of an n-dimensional Euclidean space. The main idea which permits the extension is in the author's formulation of the postulate entitled "Principle of Insufficient Reason". This postulate states that under certain transformations of the space of states of nature onto itself which leave the problem invariant, the solution is left invariant. For the Euclidean case the transformations so considered are translations. The method extends to a more general type of topological space. The exposition is concise and neat.

H. Chernoff (Stanford, Calif.).

Girshick, M. A.; Karlin, S.; and Royden, H. L. Multistage statistical decision procedures. Ann. Math. Statist. **28** (1957), 111–125.

The authors present a minimal complete class of decision procedures for a multi-stage decision problem requiring one of two decisions at each stage, where the observation at each stage is made on a random variable whose distribution belongs to the exponential family. Conditions are given which insure that these procedures, composed of monotone procedures admissible at each stage, are admissible for the multi-stage problem.

F. C. Andrews (Eugene, Ore.).

Mihalevič, V. S. Bayes solutions and optimal methods of statistical acceptance control. Ukrain. Mat. Ž. **8** (1956), 454–459. (Russian)

The author applies the results of Sobel [Ann. Math. Statist. **24** (1953), 319–337; MR **15**, 143] to a problem in statistical sampling inspection (acceptance or rejection of a lot based on the number of defective items). A sequential Bayes solution is characterized in detail. The author gives without proof results on decision functions (two possible decisions) for Poisson processes, which, he says, follow by passing to the limit (in the results on sampling inspection). According to one of these results and under hypotheses which we shall not give here, the process $x(t)$ is observed only as long as it lies between two step-functions, monotonically increasing with unit jumps.

J. Wolfowitz (Ithaca, N.Y.).

Wine, R. L.; and Freund, John E. On the enumeration of decision patterns involving n means. Ann. Math. Statist. **28** (1957), 256–259.

In an ordinary significance test of the difference be-

tween two means, say m_1 and m_2, one may decide that m_1 is significantly greater, significantly smaller, or not significantly different from m_2. With n means there are $\binom{n}{2}$ pairs of means and thus there are $3^{\binom{n}{2}}$ possible sets of pairwise comparisons. Because of order relations the admissible sets of pairwise comparisons, called decision sets, are subject to certain restrictions (a restriction equivalent to equal variances for all means seems to be implicit). If two decision sets are identical except for a permutation of the n means, they are said to exhibit the same decision pattern. A general expression for the number of decision sets is not known, but the number of decision patterns for n means is shown to be $\binom{2n}{n}/(n+1)$.

P. Meier (Baltimore, Md.).

Smith, Walter L. A note on truncation and sufficient statistics. Ann. Math. Statist. **28** (1957), 247–252.

The author proves, on the level of generality introduced into the subject by Halmos and Savage [same Ann. **20** (1949), 225–241; MR **11**, 42], that the properties of sufficiency, minimal sufficiency and completeness of a statistic are preserved under truncation of the given family of distributions. *E. L. Lehmann.*

Pitcher, T. S. Sets of measures not admitting necessity and sufficient statistics or subfields. Ann. Math. Statist. **28** (1957), 267–268.

Let X be the interval [0, 1] and let F be the σ-field of Borel sets of X. Let M be a set of probability measures on F. A σ-field $G\subset F$ is called sufficient for M if there exists a conditional expectation $E(f|G)$ of bounded F-measurable functions f which is compatible with every $m\in M$. The purpose of the paper is to give two examples of families M for which the intersection of all sufficient σ-fields is not sufficient.

In the first example, the class M is somewhat similar to the class used by Halmos and Savage [same Ann. **20** (1949), 225–241; MR **11**, 42], to show that pairwise sufficiency need not imply sufficiency.

In the second example every $m\in M$ is carried by one or two points, say x and $\varphi(x)$, of X. The function φ defined on a Borel subset of $[0, \frac{1}{2}]$ is not Borel measurable.

In the title, the word "necessity" should probably be replaced by an adjective as "minimum", "minimal", or, following an unfortunate terminology, "necessary".

L. Le Cam (Berkeley, Calif.).

Gumbel, E. J. Statistische Theorie der Ermüdungserscheinungen bei Metallen. Mitteilungsbl. Math. Statist. **8** (1956), 97–130.

Detailed instructions are given for statistical analysis of fatigue failure data by means of extreme value methods [cf. Gumbel, Nat. Bur. Standards Appl. Math. Ser. no. **33** (1954); MR **15**, 811]. For given load S the number of stress cycles N required for fracture in a tested specimen is considered as a random variable. The author considers in particular the problems of estimating the "minimum life", $N_{0,s}$, and the "endurance limit", S_0. The former is the largest value of N for which the probability of fracture at a given load S is zero. The latter is the largest value of S for which the probability of fracture is zero for every N. *D. M. Sandelius* (Göteborg).

Kitagawa, Tosio. Some contributions to the design of sample surveys. Sankhyā **17** (1956), 1–36.

This paper consists of three parts, IV–VI [for parts I–III see Sankhyā **14** (1955), 317–362; MR **16**, 1132]. In part IV exact distributions of various statistics of importance in sampling from a finite population π are derived, on the assumption that π is composed of random samples of size one from normal populations. {Reviewer's remark: Jensen [Skand. Aktuarietidskr. **35** (1952), 195–200; MR **14**, 777] obtained exact confidence limits for the mean of π for two special cases.} Part V deals with methods of controlling non-sampling errors and part VI with stratification principles. *D. M. Sandelius.*

Kudo, A. On the testing of outlying observations. Sankhyā **17** (1956), 67–76.

Given three sets of independent observations $\{x\}$: (i) n_1 from $N(m_i, \sigma^2)$, $i=1, 2, \cdots, n_1$; (ii) n_2 from $N(m^{(2)}, \sigma^2)$; and (iii) n_3 from $N(m^{(3)}, \sigma^2)$, where $N(m, \sigma^2)$ refers to a normal population with mean m and variance σ^2. One of n_1+1 possible decisions, D_i (accept H_i), is to be made where we have H_0: $m_1=m_2=\cdots=m_{n_1}=m^{(2)}$, $H_i(i\neq 0)$: $m^{(2)}=$each m_j (except m_i)$=m_i-\Delta$. A decision procedure is presented for which: $\Pr(\text{acc. } D_0|H_0)=1-p$; $\Pr(\text{acc. } D_i|H_i)$ maximized for $i\neq 0$. Other conditions which are satisfied are concerned with shifts in the m_i and invariance to linear transformations of the observations.

The optimum decision procedure involves

$$x_M=\max\{x_1, x_2, \cdots, x_i, \cdots, x_{n_1}\};$$

$\bar{x}$, the mean of samples (i) and (ii); S, the over-all standard deviation using $\bar{x}$ for (i) and (ii) and $\bar{x}_3$ for (iii). The decision rule is: select D_0 if $(x_M-\bar{x})/S\leq\lambda_p$, and select D_M if $(x_M-\bar{x})/S>\lambda_p$. Values of λ_p are to be published later.

S is replaced by σ if σ is known; in this case set (iii) is not needed, and different λ_p are needed.

R. L. Anderson (Raleigh, N.C.).

de la Garza, A. Quadratic extrapolation and a related test of hypotheses. J. Amer. Statist. Assoc. **51** (1956), 644–649.

Let $E(y_i)=\alpha+\beta x_i+\gamma x_i^2$ $(i=1, 2, \cdots, N)$, where the $\{y_i\}$ are uncorrelated and have variance V_0 and the $\{x_i\}$ are controlled, with $x_L\leq x_i\leq x_H$. In order to minimize the variance of an extrapolated value for $x=\zeta$ $(\zeta>x_H$ or $\zeta<x_L)$, only three values of x_i should be used: x_L, $\frac{1}{2}(x_L+x_H)$ and x_H. A table is given of the fraction of the N observations to be allocated to each point for selected values of $D(\zeta)=[2\zeta-(x_L+x_H)]/(x_H-x_L)$, plus the standard error of the extrapolated value.

These results are extended to the problem of testing the hypotheis that $\gamma=0$. *R. L. Anderson.*

Ozols, V. Generalization of the theorem of Gnedenko-Koroluk to three samples in the case of two one-sided boundaries. Latvijas PSR Zinātņu Akad. Vēstis 1956, no. **10** (111), 141–152. (Latvian. Russian summary)

Kiefer, J.; and Wolfowitz, J. Sequential tests of hypotheses about the mean occurrence time of a continuous parameter Poisson process. Naval Res. Logist. Quart. **3** (1956), 205–219 (1957).

This paper presents tables which give the operating characteristic and average sample time functions of sequential probability ratio tests concerning the mean of a Poisson process and analogous discrete time results for the exponential distribution. *Author's summary.*

Tsao, Chia Kuei. Approximations to the power of rank tests. Ann. Math. Statist. 28 (1957), 159–172.

Let X_{ij} ($j=1, \cdots, m_i$) be independent samples from continuous cumulative distribution functions $F_i=g_i(F)$. A method of Hoeffding [Proc. 2nd Berkeley Symposium on Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 85–92; MR 13, 479] and the reviewer [Ann. Math. Statist. 24 (1953), 23–43; MR 14, 888] permits the explicit computation of the probabilities of any ranking of the X's provided the functions g_i are polynomials. The author proposes to approximate the probabilities in the general case by approximating the functions g_i by polynomials. The method is illustrated for the case of samples from two normal distributions with sample sizes 2 and 3. The paper concludes with the evaluation of the asymptotic efficiency of certain rank sum tests proposed by the author earlier [ibid 26 (1955), 94–104; MR 16, 941]. *E. L. Lehmann* (Berkeley, Calif.).

Bahadur, R. R.; and Savage, Leonard J. The nonexistence of certain statistical procedures in nonparametric problems. Ann. Math. Statist. 27 (1956), 1115–1122.

Let $X_1, X_2, \cdots$ (real) be independent and identically distributed according to some distribution function F in $\mathscr{F}$. Let δ be any closed sequential procedure for estimating the mean μ_F of F by a confidence interval of length $\leqq c$. It is clear that, if $\mathscr{F}$ is sufficiently rich, for $\varepsilon>0$ there will be an integer N and a subclass $\mathscr{F}_1$ of $\mathscr{F}$ such that δ requires $\leqq N$ observations with probability $\geqq 1-\varepsilon$ for each F in $\mathscr{F}_1$, such that the probabilities of any subset of R^N under any two members of $\mathscr{F}_1$ differ by $<\varepsilon$, and such that $\{\mu_F, F \text{ in } \mathscr{F}_1\}$ is an unbounded set. It follows at once that the probability of coverage by the confidence interval has infimum zero. The authors give sufficient conditions on $\mathscr{F}$ to make it sufficiently rich for this result to hold, and treat similarly the problem of estimating F by confidence bands which are themselves d.f.'s. *J. Kiefer* (Ithaca, N.Y.).

Sukhatme, Balkrishna V. On certain two-sample nonparametric tests for variances. Ann. Math. Statist. 28 (1957), 188–194.

A new test is proposed for testing the equality of two distribution functions, F and G, where the alternatives are only a result of differing scale parameters. The mean and variance of the test statistic is computed under the hypothesis tested and the alternatives. The test is proved to be consistent. The author shows that the asymptotic variance of a statistic proposed by Lehmann [same Ann. 22 (1951), 165–179; MR 12, 726] depends upon the form of the distributions and hence the corresponding test is not distribution free. Asymptotic relative efficiencies of a test of Mood [ibid. 25 (1954), 514–522; MR 16, 154] and the proposed test with respect to the classical variance ratio F test are computed. *F. C. Andrews.*

Weiss, Lionel. A certain class of tests of fit. Ann. Math. Statist. 27 (1956), 1165–1170.

Let $y_1 \leqq y_2 \leqq \cdots \leqq y_n$ be the order statistics corresponding to a sample of size n on a random variable X having density function f with $\int_0^1 f(x)dx=1$. Defining

$$w_1=y_1, \cdots, w_i=y_i-y_{i-1}, \cdots, w_{n+1}=1-y_n,$$

let $Z_1 \leqq Z_2 \leqq \cdots \leqq Z_n$ be the order statistics corresponding to the w's. This note proposes a test of the hypothesis $f(x)=1$ based upon the statistic $\sum_{i=1}^{n+1} Z_i^u$, $u>1$. The joint distribution of the Z's is obtained and the consistency of the test against a wide class of alternatives is proved. *F. C. Andrews* (Lincoln, Nebr.).

Weiss, Lionel. On the uniqueness of Wald sequential tests. Ann. Math. Statist. 27 (1956), 1178–1181.

Under certain mild restrictions on the distributions involved, it is shown that the probabilities of the two types of error uniquely determine the two bounds characterizing the Wald sequential probability ratio test. (Author's summary.) *H. A. David* (Melbourne).

James-Levy, G. E. A nomogram for the integral law of Student's distribution. Teor. Veroyatnost. i Primenen. 1 (1956), 272–274 (1 plate). (Russian. English summary)

See also: Čentson, p. 831; Batchelor, p. 843; Zabel, p. 859.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Marsicano, Fènix Roberto. The Poinsot cones corresponding to the movement of the cross-piece of the Cardan joint. Atti Sem. Mat. Fis. Univ. Modena 7 (1953–54), 22–27 (1956). (Spanish)

This note supplies an analytic interpretation of some aspects of the graphical method of Carrizo Rueda [Ciencia y Técnica 115 (1950), 74–88; MR 12, 293], for the purpose of determining the speed ratio relevant to a Cardan coupling (universal joint). *F. B. Hildebrand.*

Jarre, Gianni. Proprietà dinamiche dei regolatori meccanici di velocità. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 90 (1955–56), 552–562.

This note subdivides a class of mechanical speed regulators according to properties of a quadratic form representing the relevant kinetic energy. The usual methods of stability analysis are then outlined and applied to several specific examples.
F. B. Hildebrand (Cambridge, Mass.)

Sevin, Eugene. Min-max solutions for the linear mass-spring system. J. Appl. Mech. 24 (1957), 131–136.

For the solutions of $m\ddot{x}+m\omega^2x=f(t)$ with $x=0$ and $\dot{x}=0$ at $t=0$, the absolute upper and lower bounds for the maximum value of x are considered. It is assumed that the forcing function $f(t)$ is non-negative, has a duration t_1, and an integral from 0 to t_1 or total impulse equal to I. The upper bound, $I/m\omega$, results from $f(t)$ being an impulsive type load I applied at one single instant. The lower bound always results from an initial and final impulsive type load, and an intermediate constant force. The ratios of these, and the form of the bound, depend on whether $\omega t_1>5\pi/6$, $\omega t_1<2\pi/3$, or ωt_1 lies between these two values. *P. Franklin* (Cambridge, Mass.).

Albrecht, Rudolf. Ein Satz über Massenmomente n-ter Ordnung. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. 1956, 125–128 (1957).

For an arc of an orbit under a central force, a geometric theorem due to Laisant results from a combination of the law of sectorial areas and the relation of the center of

mass of a sector to that of its arc, suitably weighted. This note generalizes the latter relation by considering mean points based on moments of higher order, and dimensions higher than two. *P. Franklin* (Cambridge, Mass.).

Laurenti, Fernando. Sopra una superficie di sesto ordine che si presenta in Meccanica. Atti Sem. Mat. Fis. Univ. Modena 7 (1953–54), 147–166 (1956).

Consider the motion of a heavy rigid body about a fixed point O, and consider the locus S described by the endpoint P of the moment vector of the impulse of the body with respect to O. The so-called second impulse theorem provides the vector equation of motion of the body for which two integrals are deduced. Using these two integrals the author shows that the locus S is a surface of degree six with three double points situated on the line which joins the fixed point O and the center of mass G of the body. Much space is devoted to the study of the double points of the surface S. *E. Leimanis.*

Faure, Robert. Vibrations non linéaires: action asynchrone, cas du phénomène Bethenod. C. R. Acad. Sci. Paris 243 (1956), 1824–1827.

The system

$$L(\theta)i'+L'(\theta)\theta'i+Zi=e(t),\ M\theta''+D\theta'+C\theta=\tfrac{1}{2}L'(\theta)i^2,$$

where Z, M, D, C are positive constants and $e(t)$ has period $T=2\pi/\omega$, is shown to have a solution of period T under the assumptions: $L(\theta)=L_0+L_1\theta+L_2\theta^2+\cdots$ (for small θ) with $L_0>0$, $e(t)=\sum \alpha_n \exp(in\,\omega t)$ with $\alpha_0=0$ and $\sum|\alpha_n|=\eta<\infty$, and either η sufficiently small or ω sufficiently large. Proof by successive approximations.

G. E. H. Reuter (Manchester).

Grega, B. The movement of the padding machine's looper. Acta Tech. Acad. Sci. Hungar. 16 (1957), 219–231. (German, French, and Russian summaries)

Venkatesan, N. S. On the solutiono f the equations of internal ballistics with a cubic form function. Proc. Nat. Inst. Sci. India. Part. A. 22 (1956), 129–136.

The author asserts "that for all propellant shapes that one comes across in practice, the form function can be taken as $(1-f)(1+\theta f+\theta_1 f^2)$, where f is the fraction of the web size remaining unburnt at any instant, θ, θ_1, being constants of the powder." This statement is supported by asserted formulas for grains which are cylindrical, tubular, cubical, or spherical, respectively. (This evidence does not include consideration of the multiperforate grain used in most of the United States artillery weapons.) Accepting the "fundamental equations of Internal Ballistics" in the form given by Corner [Theory of the interior ballistics of guns, Wiley, New York, 1950; MR 12, 213], the author obtains some explicit but complicated relations, which for numerical data should yield the maximum pressure. With some simplifying assumptions, but retaining a covolume correction term, the author computes the muzzle velocity, assumed to occur after all-burnt. Here are given no numerical tables and no discussion of the physics or physical chemistry of the propellant gases. Rather intricate mathematical formulas are here computed on the relatively uncertain basis of theoretical formulas and the unhesitating acceptance as fact of the simultaneous instantaneous ignition of all powder grains and their subsequent symmetrical burning according to elementary geometrical principles. The reader must consult "Internal ballistics" [Ministry of Supply, London, 1951] to understand the symbols used.

A. A. Bennett (Providence, R.I.).

Marsicano, Fenix R. Dynamics of material systems with variable mass. An. Soc. Ci. Argentina 162 (1956), 8–13. (Spanish)

The author starts with the vectorial differential equation of movement of a material particle whose mass varies with time alone, in terms of partial derivatives of the kinetic energy function. Extending the method to an arbitrary holonomic system, he obtains an equation not found in the works of the various authors which he cites.

A. A. Bennett (Providence, R.I.).

Jarmolow, Kenneth. Dynamics of a spinning rocket with varying inertia and applied moment. J. Appl. Phys. 28 (1957), 308–313.

Tawakley, V. B. Internal ballistics of composite charge when they burn according to the geometric form functions for spheres. Proc. Nat. Inst. Sci. India. Part A. 22 (1956), 54–62.

The author derives closed form solutions (of complicated mathematical type) for the case of a composite charge consisting of two types of powder, each burning uniformly according to the geometric law for spheres (or cubes). Linear rates of burning and a constant temperature of propellant gasses are assumed. The exponent γ for adiabatic burning is assumed to be independent of the choice of charge. No comparison with experiment is indicated. *A. A. Bennett* (Providence, R.I.).

See also: Craig, p. 798; Cuming, p. 805; Wintner, p. 805; Fesenkov, p. 857.

Statistical Mechanics

Green, Melville S. Boltzmann equation from the statistical mechanical point of view. J. Chem. Phys. 25 (1956), 836–855.

This important paper is devoted to the fundamental problems of deriving the Boltzmann equation for a dilute classical gas from the mechanical equation of motion (Liouville equation) and of determining additional corrections to it by inclusion of triple and higher order collisions. The author starts from the usual hierarchy of integro-differential equations following from the Liouville equation for the distribution functions for 1, 2, 3, $\cdots$ molecules. His first basic contribution is to have found an explicit solution of this hierarchy in power series in the density. This solution is expressed in terms of the solutions of the Liouville equation for groups of 2, 3, $\cdots$ molecules. The derivation is based on an Ursell-Mayer expansion similar to the algorithm used for the equilibrium properties of dilute gases.

The author then proceeds with a long and complicated analysis tending to show that for sufficiently large times the n-particle distribution functions ($n=1, 2, 3, \cdots$), taken for connected configurations of the particles, can be entirely expressed in terms of the values at the same instant of a newly defined one-particle distribution function $\bar{\psi}_1$. Connected configurations are defined as configurations where the particles are not located in two or more widely separated groups. The definition of $\bar{\psi}_1$ depends on the precise choice made for this concept of connectedness.

Knowing that, subject to certain restrictions, the n-particle distribution functions φ_n at time t are functionals of $\bar{\varphi}_1$ at the same time t, one expects in particular that $\bar{\varphi}_1$ can be expressed at each instant in terms of the true one-particle distribution φ_1. Each φ_n becomes in turn a functional of φ_1 at the same time, and one thus expects that the first equation in the hierarchy will express the rate of change of φ_1 in terms of φ_1 itself. Proceeding along these lines the author indeed obtains in lowest order the well-known form of the Boltzmann equation. Going to the next order he is also able to calculate the triple collision term. *L. Van Hove* (Utrecht).

Brinkman, H. C. On Kramer's general theory of Brownian motion. Physica 23 (1957), 82–88.

This paper considers Kramer's general equation for Brownian motion in phase space [Physica **7** (1940), 284–304; MR **2**, 140] and discusses under which conditions a natural (approximate) expression for the free energy of the Brownian particles decreases with time. These conditions can be simply expressed in terms of the moments of the (random) momentum transferred to the Brownian particles by the fluctuating force. They are fulfilled for the usual forms adopted for these moments as well as for the form obtained for a simplified model of Brownian motion. *L. Van Hove* (Utrecht).

Bloch, F. Generalized theory of relaxation. Phys. Rev. (2) **105** (1957), 1206–1222.

This paper contains a very general theory of relaxation phenomena, and a specific application to nuclear spin relaxation in strong time-dependent magnetic fields. The theory applies to any system governed by a time-dependent Hamiltonian and weakly coupled to a molecular thermal reservoir; the only limitation on the theory being that the interaction is weak compared to the average internal frequencies of the molecular system. The usefulness of the formalism is demonstrated for nuclear spins in rotating magnetic fields, although no account is taken of spin-spin coupling. *P. W. Anderson.*

Katsura, Shigetoshi. On the Bose-Einstein condensation. Progr. Theoret. Phys. **16** (1956), 589–603.

The author shows that the canonical partition function of the ideal Bose-Einstein gas can be evaluated using a method of steepest descent even at temperatures below the critical temperature. For the condensed state, the path of integration is chosen so as to form a cusp at a singularity, whereas for the uncondensed state the path is chosen through a saddle point. This conclusion was announced also by Ford and Berlin [Phys. Rev. (2) **99** (1955), 638].

The paper also gives a sufficient condition for the validity of using cluster integrals evaluated in the limit $V\to\infty$ for the computation of the canonical function of the imperfect gas. *G. Newell* (Providence, R.I.).

Monin, A. S. A statistical interpretation of the scattering of microparticles. Teor. Veroyatnost. i Primenen. **1** (1956), 328–343. (Russian. English summary)

The kinetic theory of microparticles moving in a dispersive and absorptive medium is constructed on the supposition that along a random path the radius vector of the particle, the unit vector of its velocity and its energy taken together form a vector Markov stochastic process. The processes of scattering in a homogeneous and isotropic medium are considered in detail when there are no energy changes. A solution to the kinetic equation is found for these processes for the case of an instantaneous point source of particles. (Author's summary.) *D. Falkoff.*

Crowe, C. M. A kinetic model for diffusion of gases in polymers. Trans. Faraday Soc. **53** (1957), 692–699.

Huxley, L. G. H. Free path formulae for the coefficient of diffusion and velocity of drift of electrons in gases. Austral. J. Phys. **10** (1957), 118–129.

Free path methods are used to derive formulae for the coefficient of diffusion and the drift velocity of electrons in weakly ionized gases in the general case in which the collisional cross section is a function of the speed of an electron and the law of scattering of electrons in single collisions with molecules is not restricted to a few specific cases. It is found that the results are, in effect, the same as those derived by means of the methods of Maxwell and Boltzmann. *From the author's summary.*

Huang, Kerson; Yang, C. N.; and Luttinger, J. M. Imperfect Bose gas with hard-sphere interaction. Phys. Rev. (2) **105** (1957), 776–784.

The quantum theory of a system of hard spheres has been treated approximately in a previous paper by the first two authors [Phys. Rev. (2) **105** (1956), 767–775; MR **18**, 702] by the so-called pseudo-potential method which consists in replacing the hard sphere interaction by an approximate contact interaction of δ-function type. This pseudo-potential gives results correct to second order in the radius of the hard spheres. It is here applied to the calculation of the virial coefficients of a Bose gas with hard sphere interaction. For this case the dimensionless expansion parameter turns out to be the ratio of hard sphere radius to thermal de Broglie wavelength. The results are valid to second order in this parameter; they are essentially low temperature results. The calculation of the virial coefficients is based on the usual fugacity expansion. The last section of the paper studies the Bose condensation for a fictitious Bose gas whose energy levels are the energy levels of a hard sphere gas taken to first order in the hard sphere radius. This set of energy levels is not compatible with the stability requirements which are fulfilled for realistic systems. They do however give an acceptable equation of state by application of the Maxwell rule and the corresponding isotherms show in contrast to the ideal Bose gas a first order transition where for each temperature the pressure remains constant over a finite density interval. *L. Van Hove* (Utrecht).

Hiroike, Kazuo. Radial distribution function of fluids. I. J. Phys. Soc. Japan **12** (1957), 326–334.

Several approximation methods for obtaining the radial distribution function are examined by studying whether the relationship

$$\frac{\partial}{\partial T}\left(\frac{p}{T}\right)=\frac{\partial}{\partial V}\left(\frac{E}{T^2}\right)$$

is satisfied in each method. *C. N. Yang.*

★ **Montroll, Elliott W. Theory of the vibration of simple cubic lattices with nearest neighbor interactions.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, pp. 209–246. University of California Press, Berkeley and Los Angeles, 1956. $6.25.

The vibrations of a cubic lattice with 1, 2, 3, and n

dimensions where n is large is considered. Only the forces between nearest neighbors are taken into account and it is assumed that the relative variation in interparticle distance during a motion is so small that these forces are linear in the relative displacement. Under these assumptions the motions in each of the directions parallel to the lattice are independent and the author, therefore, restricts himself to motions in the 1 direction. Tensor forces are allowed so that the adjacent particles in the perpendicular direction, say the 2 direction, exert shear forces on each other but these shear forces are smaller than those exerted by adjacent particles in the 1 direction.

The normal modes for these motions are set up and the set of possible frequencies is calculated. The frequencies extend from zero up to a limiting value ν_L corresponding to a motion where any two adjacent particles are moving in opposite directions. The limiting form of this frequency spectrum is found as the number of particles in the lattice approaches infinity by the method of the characteristic function, and is explicitely evaluated for 1, 2, 3 and a large number of dimensions. The spectrum has a singularity at the maximum frequency for a one-dimensional lattice. In the two-dimensional case there are singularities at two frequencies which would be maximum rfequencies in the absence either of the couplings of adjacent particles in the 1 or 2 direction. In the three-dimensional case there are no singularities, but the spectrum has corners at these particular frequencies. For n large the spectrum is Gaussian with a half width proportional to n and centered about half the maximum frequency.

The distribution of an arbitrary particle in space under thermal fluctuations at a temperature T is computed from the Slater sum and found to be Gaussian. The dispersion is calculated in the limit of high and low temperature $h\nu_L/kT \ll 1$ and $h\nu_L/kT \gg 1$. For one dimension the dispersion can be quite large compared to interparticle distance so that particles cannot be considered as localized at any temperature. For the two-dimensional lattice the particles are localized at low temperatures but not at high temperatures. In the three-dimensional lattice they are localized at all temperatures. The localization is due to the shear forces and it is made clear that as the shear forces go to zero the particles lose their localization.

A method is sketched for treating the non-ideal lattice; that is, one with an occasional hole or impurity.

R. Kulsrud.

See also: Popoff, p. 850; Yaglom, p. 851; Kanazawa, p. 852; Höhler, p. 852; Seiden, p. 854; Placzek, p. 854.

Elasticity, Visco-elasticity, Plasticity

Aržanyh, I. S. Dynamical potentials of the theory of elasticity. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 13 (1954), 3–17. (Russian)

The system of partial differential equations for the displacement vector $u=u(x, y, z, t)$ in the dynamical theory of elasticity is

$$(\alpha \text{ grad div} - \beta \text{ rot rot} - \partial^2/\partial t^2)u = f,$$

where α and β are elastic constants. After giving a general formula for the representation of solutions of this equation, and determining explicitly a "fundamental solution", the author proves a formula which is analogous to the formula of Kirchhoff in the theory of the wave equation. The various "retarded potentials" occurring in this "generalized Kirchhoff formula" are then analyzed, and integro-differential equations for the solution (displacement vector) of the first and second boundary value problems are deduced.

J. B. Diaz.

Hanin, Meir; and Reiner, Markus. On isotropic tensor-functions and the measure of deformation. Z. Angew. Math. Phys. 7 (1956), 377–393.

The formal power series $y=\sum_0^\infty \mathscr{F}_n x^n$ in the 3×3 matrix x, where $\mathscr{F}_n$ is a power series in the principal invariants I, II, III of x, may be rearranged in the form $y=F_0 1+F_1 x+F_2 x^2$. The authors consider the problem of evaluating the three F_i in terms of the given $\mathscr{F}_n$. They succeed in doing this explicitly when the $\mathscr{F}_n$ are constants. The result is applied to expressing in terms of one another some of the familiar strain measures used in finite elasticity theory.

C. Truesdell (Bloomington, Ind.).

Bressan, Aldo. Sulla possibilità di stabilire limitazioni inferiori per le componenti intrinseche del tensore degli sforzi in coordinate generali. Rend. Sem. Mat. Univ. Padova 26 (1956), 139–147.

The author sets up the mean value theorem of Signorini [Ann. Scuola Norm. Sup. Pisa (2) 2 (1933), 231–251]; he uses general curvilinear coordinates, with indices based on a fixed set of Cartesian coordinates. Signorini's theorem may be regarded as an expression in terms of assigned quantities for the mean of the scalar $t^{km}\phi_{k,m}$, where the vector ϕ_k is arbitrary. When the stress tensor t^{km} is symmetric, this sum is $t^{km}\phi_{(k,m)}$, which may be written as $\sum_{a=1}^6 t_a f_a$ with $f_1=\phi_{1,1}, \cdots, f_4=\phi_{(1,2)}, \cdots$. If ϕ_k is so selected that $f_a=0$ for $a\neq b$, then the sum reduces to the one term $t_b f_b$. Since $\overline{t_b f_b} \leq \overline{|t_b|}\cdot\overline{|f_b|}$, a general lower bound for the greatest stress $|t_b|_{\max}$ results for each such choice of the vector ϕ_k. The author finds a solution in cylindrical coordinates and evaluates the lower bounds explicitly.

C. Truesdell (Bloomington, Ind.).

Angelitch, T. P. Eine Bemerkung zu den Gleichungen von Beltrami-Michell. Acad. Serbe Sci. Publ. Inst. Math. 9 (1956), 93–94.

The author corrects an error in a previous paper [same Publ. 5 (1953), 1–4; MR 15, 578].

C. Truesdell (Bloomington, Ind.).

Capriz, Gianfranco. Sopra le deformazioni elastiche finite di un solido tubolare. Rend. Mat. e Appl. (5) 15 (1956), 228–262.

For many years Signorini and his school have emphasized a certain incompatibility between the classical linear theory of elasticity and the classical theory of finite elastic strain. This incompatibility consists in the fact that there are soluble boundary problems in the linear theory which do not emerge as limit cases of soluble problems of the finite theory. The tool used is a formal power series in a perturbation parameter. The system for the leading terms coincides with that for the linear theory; there are cases when this system is soluble but none of the higher order systems is compatible. In this paper the author sets up the corresponding problem for the elastic line. He concludes that in the incompatible case the linearized theory gives a first approximation to the solution of a dynamic rather than a static problem from the finite theory.

C. Truesdell (Bloomington, Ind.).

Horvay, G. Some aspects of Saint Venant's principle. J. Mech. Phys. Solids 5 (1957), 77–94.

The author is concerned with the rates of decay appropriate to the stresses and displacements induced by various special self-equilibrated distributions of normal and shearing tractions applied to (a) the transverse edge of a semi-infinite strip, (b) one side of the boundary of a rectangular domain, and (c) a finite segment of the boundary of a half-plane. The paper aims primarily at a compilation and discussion of approximate results established elsewhere by the present author and J. S. Born with the aid of a variational scheme in the two-dimensional theory of elasticity.

In view of the title of the paper, and because of the troubled history of Saint-Venant's principle, a precise statement of the particular version of the principle which the author intended to support would have been helpful. Clearly, the rates of stress-decay associated with self-equilibrated loadings are relevant to a meaningful statement of Saint-Venant's principle only in comparison with the corresponding rates of decay characteristic of loadings which are not self-equilibrated. Yet, there exist examples of wedge-shaped domains in which self-equilibrated surface tractions may give rise to stresses which decay less rapidly than do those stemming from loadings with a non-vanishing resultant. In the light of this observation, the general statements at the beginning of the paper, regarding arbitrary simply connected regions, are open to misinterpretation. *E. Sternberg* (Providence, R.I.).

Williams, M. L. On the stress distribution at the base of a stationary crack. J. Appl. Mech. 24 (1957), 109–114.

This paper is concerned with the generalized plane-stress solution appropriate to a semi-infinite rectilinear crack in a homogeneous and isotropic elastic sheet of infinite extent. The two faces of the crack are assumed to be free from surface tractions. Both the symmetric loading case (extension) and the antisymmetric case (bending) are treated. The latter case has apparently not received previous analytic attention. The solutions obtained are in closed form, and are established with the aid of well-known elementary Airy functions. The chief contribution of the present paper to the existing literature on the subject at hand lies in the extensive discussion of the ensuing stress and strain-energy distribution around the root of the crack. The results seem to be of considerable interest in connection with failure speculations.

E. Sternberg (Providence, R.I.).

Verma, P. D. S. Hypo-elastic pure flexure. Proc. Indian Acad. Sci. Sect. A. 44 (1956), 185–193.

The author considers an incompressible hypo-elastic body of grade zero. In cylindrical polar co-ordinates, he takes all shear stresses as zero and all normal stresses as functions of r only, and he takes the longitudinal velocity w as zero. He shows that the only solution of this type is given by

$$S_r{}^r=\frac{a^2}{2r^2}-\frac{1}{2},\ S_\theta{}^\theta=-\frac{1}{2},\ S_z{}^z=0,$$

$$u=\frac{a}{r}\left(\frac{\mu}{\rho}\right)^{1/2},\ v=0.$$

This represents pure bending of a cylinder whose face $r=a$ is free of traction. To maintain the motion, not only a bending couple but also a certain traction on the face $r=b$ must be applied. The author finds that the stresses are much the same as those obtained by Seth from a certain theory of finite elastic deformation.

C. Truesdell (Bloomington, Ind.).

Basilevich, V. Shearing stress in bending of T beams. Acad. Serbe Sci. Publ. Inst. Math. 9 (1956), 59–67.

The solution of the problem of bending of prismatic cantilever can be reduced to the determination of the stress function $\phi(x, y)$ which satisfies the differential equation in the region of the cross-section $\Delta\phi=0$, and the condition

$$\frac{\partial\phi}{\partial y}-\frac{P}{2J}x^2+\frac{\mu}{1+\mu}\frac{P}{2J}y^2\cos(nx)-\frac{\partial\phi}{\partial x}\cos(ny)=0$$

on the boundary (where Δ is the Laplacian operator). Here P is the force applied at the end of the beam, J the moment of inertia of the cross-section, μ Poisson's ratio, n the normal to the boundary, (x, y) the principal axis of the cross-section.

The calculation of the shearing stresses depends on the form of cross-section and must be found in all special cases of the boundaries. This paper presents the solution of the problem in the case of the T-cross-section.

R. Gran Olsson (Trondheim).

Baldacci, Riccardo F. Sulla integrazione diretta del problema di Saint-Venant in termini di tensioni. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 90 (1955–56), 604–610.

The author treats the problem of Saint Venant for a cylinder with free sides in tensor calculus assuming that the unknown functions are special stress components. The three undetermined stress equations of equilibrium with five Beltrami's equations of compatibility and boundary conditions show that the normal tension (σ_z) can be expressed as a linear function of x, y, z with six constants and that the shearing stresses τ_x, τ_y are functions only of x, y. Then the problem is reduced to the plane one with three equations and two unknowns. Using a special transformation one obtains a system of two equations of the first order, which satisfy the relations div $\tau^0=0$, rot $\tau^0=0$ because of the existing harmonic stress function. The boundary condition on the contour represents a special case of Neumann's problem which gives an integral relation among the constants of the normal tension from which they may be determined.

D. P. Rašković (Belgrade).

Szabó, J. Application of the matrix theory to the calculation of continuous beams. Acta Tech. Acad. Sci. Hungar. 16 (1957), 175–193. (German, French and Russian summaries)

Vodička, Václav. Biegungsschwingungen in zusammengesetzten Stäben. Z. Angew. Math. Mech. 37 (1957), 44–51. (English, French and Russian summaries)

Barta, J. Quelques formules pratiques pour le calcul statique des poutres Langer. Acta Tech. Acad. Sci. Hungar. 16 (1957), 407–414. (German, English, and Russian summaries)

Buchwald, T.; and Tiffen, R. Boundary-value problems of simply-supported elastic plates. Quart. J. Mech. Appl. Math. 9 (1956), 489–498.

This paper is a continuation of Mr. Tiffen's work which has concerned itself with the solution of boundary

value problems in a two-dimensional elastic medium by means of complex potentials [same Quart. **5** (1952), 237–252, 344–351, 352–360; **6** (1953), 344–369; MR **14**, 110, 333; **15**, 267] and the application of these methods to problems of thin elastic plates [ibid. **8** (1955), 237–250; MR **16**, 1177], using the notation of A. C. Stevenson [Phil. Mag. (7) **23** (1942), 639–661; MR **4**, 63]. Complementary solutions are obtained for two different formulations of simple support for the infinite half plane. These are combined with a singular solution to represent an isolated load in a simply-supported half plane. Additional complementary solutions are given which can be used to compensate for deflections and bending moments along a second edge, $y=a$, for the cases where these edge values are or are not expressible as Fourier integrals. These solutions are combined with the previous ones to solve the simply-supported infinite strip under a concentrated load. *F. T. Geyling* (Murray Hill, N.J.).

Kempner, Joseph; Pandalai, K. A. V.; Patel, Shared A.; and Crouzet-Pascal, Jacques. Postbuckling behavior of circular cylindrical shells under hydrostatic pressure. J. Aero. Sci. **24** (1957), 253–264.

The postbuckling behavior of initially perfect, thin-walled, circular cylindrical shells under hydrostatic pressure is investigated with the aid of the principle of stationary potential energy together with appropriate approximate deflection functions. Calculations show that postbuckling equilibrium configurations exist for loads greater than as well as loads slightly less than the critical load calculated from small-deflection theory. Loads less than the critical load are obtained only for a finite range of a parameter indicative of shell geometry. For loads corresponding to radial displacements of the order of the shell thickness, it is found that the number of circumferential waves remains essentially constant with increasing deflection and equal to the number of waves developed at buckling. *Author's summary.*

Visarion, V. Une méthode générale de résolution des couvertures élastiques minces. Com. Acad. R. P. Romîne **6** (1956), 635–640. (Romanian. Russian and French summaries)

In the present paper the author generalizes some results obtained previously by A. L. Goldenvaiser [Theory of thin elastic shells, Gostehizdat, Moscow, 1953; Prikl. Mat. Meh. **8** (1944), 441–467; MR **16**, 645; **7**, 42]. Subject to certain conditions, the author obtains a system of ordinary differential equations which are neither integrated nor interpreted properly. *K. Bhagwandin.*

Seide, Paul. Axisymmetrical buckling of circular cones under axial compression. J. Appl. Mech. **23** (1956), 625–628.

To determine the influence of taper, the axisymmetric mode of buckling of a right circular cone is studied. The approximate analysis introduced for cylinders and spheres by Timoshenko [Theory of elastic stability, McGraw-Hill, New York, 1936, pp. 419–497] is used. For the case of a constant thickness cone, the resulting simultaneous equations for the displacement can be solved in terms of Bessel functions. The stability determinant is a highly complicated combination of Bessel functions which simplifies considerably when Poisson's ratio vanishes. The author notes the usefulness of this case by observing that most buckling loads are not sensitive to Poisson's ratio. A solution is found for large values of a parameter which states that the buckling load is the same as that of a cylinder having a thickness equal to the projection of the cone thickness on a plane perpendicular to the longitudinal axis. An approximate formula for long cones is suggested in analogy with that of Batdorf, Schildcrout, and Stein [NACA Tech. note no. **1343** (1947)] for long cylinders. *G. H. Handelman* (Troy, N.Y.).

Naghdi, P. M. On the theory of thin elastic shells. Quart. Appl. Math. **14** (1957), 369–380.

The author derives an approximate system of stress strain relations for the linear theory of thin, elastic, isotropic shells, such that the effects of transverse shear deformation and transverse normal stress deformation are taken into account. The results and the method of derivation are generalisations of earlier developments along the same lines for the case of rotationally symmetric deformations of shells of revolution. [E. Reissner, J. Math. Phys. **31** (1952), 109–119; MR **13**, 1006]. Of particular interest in the present paper is the way in which account is taken of the fact that shear stress resultants and couples are such that $N_{12}\neq N_{21}$ and $M_{12}\neq M_{21}$ while at the same time one has $\tau_{12}=\tau_{21}$ for shear stresses. This problem did not arise in the earlier treatment of symmetrical deformations of shells of revolution. *E. Reissner.*

Verma, G. R. Application of Dirac's delta-function in isolated force problems of semi-infinite elastic solid of isotropic and non-isotropic materials. Z. Angew. Math. Mech. **37** (1957), 34–38. (English, French and Russian summaries)

The well-known solutions corresponding to a concentrated load acting normal to a homogeneous elastic half-space, which is isotropic or transversely isotropic, are re-established with the aid of Dirac's delta function. *E. Sternberg* (Providence, R.I.).

Miles, John W. Supersonic flutter of a cylindrical shell. J. Aero. Sci. **24** (1957), 107–118.

The author continues his theoretical study of the panel flutter problem by considering the flutter of a uniform, infinitely long, thin cylinder moving in the axial direction with supersonic speed. Flutter in this instance is represented by undamped travelling-wave motion of the shell in a generally helical pattern.

Attention is restricted to shell motions of relatively small wave-length (on the order of the geometric mean of the shell radius and thickness), thus permitting neglect of the effects of the shell curvature when evaluating the aerodynamic forces. Thus, the aerodynamic pressures on both sides of the shell are evaluated on the basis of small-perturbation theory for travelling-waves on a flat plate. It is presumed that the shell is filled internally with a fluid of arbitrary properties, the fluid being essentially at rest and incompressible.

To describe the structural performance of the shell, the conventional small-displacement equations are employed in the form used for the linear study of axial shell buckling.

The flutter stability boundaries are studied in elegant fashion by means of Cauchy-Nyquist arguments. Theoretical curves are deduced for the shell thickness required to avoid panel flutter during a given flight condition, and it is pointed out that an increase of structural damping will not stabilize the flutter instabilities present in the basic system.

Earlier work by Leonard and Hedgepeth [NACA Tech.

Note no. **3638** (1956)] treats the same problem, but concentrates on longer wave-length. Author claims his analysis is the more realistic, although the arguments do not appear to the reviewer to be final. *M. Goland.*

Jones, J. P. Helicopter rotor blade flapping and bending: Part II: Elastic blades. Aicraft Engrg. **29** (1957), 107–112.

Ferrer Figueras, Lorenzo. Stationary movement of a thread with respect to a system of axes rotating about the *OZ* axis with uniform acceleration. Collect. Math. **8** (1955–1956), 109–169. (Spanish)

The author first derives the differential equations which govern the stationary motion of a perfectly flexible and inextensible string, referred to a set of axes rotating about a fixed axis with uniform acceleration. A number of special solvable problems are then treated explicitly and a comparison is made of the applicability of the analytical methods of Appell and Terradas for this purpose.
F. B. Hildebrand (Cambridge, Mass.).

Heinrich, G. Schwingungen durchströmter Rohre. Z. Angew. Math. Mech. **36** (1956), 417–427. (English, French and Russian summaries)

Pisarenko, G. S. Transversal vibrations of a turbine blade of variable cross-section in the absence of rim-constraints. Akad. Nauk Ukrain. RSR. Prikl. Meh. **1** (1955), 67–82. (Ukrainian Russian summary)

The present work is a continuation and generalization of a previous contribution [Inžen. Sb. **5** (1948), no. 1, 108–132; MR **13**, 800]. By means of variational principles the author obtains a non-linear integro-partial differential equation of the fourth order in the variables x and t. The boundary-conditions are

$$y(0,t)=y'(0,t)=y''(l,t)=y'''(l,t)=0;$$

l denotes the length of the blade in question. The author obtains a second-order approximation for the determination of the dissipation energy: polynomial approximations are also introduced. Finally the author illustrates his theory by solving a specific problem. Resonance curves as well as the eigen-functions are obtained. The author's analysis seems reasonable in the case of weak non-linearity. *K. Bhagwandin* (Oslo).

Alekseev, A. S.; and Cepelev, N. V. Intensity of reflected waves in a stratified anisotropic elastic medium. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 1021–1035. (Russian)

The present study of the intensity distribution of reflected waves in stratified anisotropic media is mostly of an expository nature. As a matter of fact the exposition lacks mathematical rigour. Initial- and boundary-value problems are not considered, and, consequently, it becomes rather difficult to establish properly the validity of the type of formulae. The authors do, however, present some application of their theory, but without any numerical data. *K. Bhagwandin* (Oslo).

Davin, Marcel. Sur la vibration forcée d'un sol stratifié. C. R. Acad. Sci. Paris **243** (1956), 565–567.

Continuing the work reported in a previous note [same C. R. **243** (1956), 352–354; MR **18**, 85] the author extends the theory of Burmister concerning deformations of a medium consisting of a finite layer over a half-space. The extension is that from the case of static to dynamic loads. The author's previous note had been confined to penetrating waves. In the present note the case of waves returning upward through the finite layer is considered. Such waves exist only with wave-lengths less than the product of the velocity of propagation by the period of the disturbance impressed on the layer. The author notes various special cases, including that of Rayleigh waves.
A. Blake (Ann Arbor, Mich.).

Ferrarese, Giorgio. Sulle oscillazioni non linearizzate dei fili perfettamente elastici. Rend. Mat. e Appl. (5) **15** (1956), 263–270.

For a certain non-linear theory of motion of an elastic cord, the author sets up a series solution in powers of a parameter, as introduced by Signorini for the general theory. For the plane case, he identifies the successive terms as representing transversal and longitudinal oscillations. *C. Truesdell* (Bloomington, Ind.).

Sentis, André. Sur la propagation des ondes dans un milieu visco-élastique. C. R. Acad. Sci. Paris **244** (1957), 558–560.

The author determines the velocity of propagation of distortion and dilatation waves in a special linear visco-elastic medium of the Voigt-Kelvin type. {Reviewer's remark: The author seems to be unaware of the large literature on this subject.} *W. Noll.*

Aržanyh, I. S. Regular integral equations of the dynamics of an elastic body. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **15** (1955), 79–85. (Russian)

Consider an isotropic three-dimensional elastic body with density ρ and Lamé's constants λ and μ, which occupies an open bounded set Q whose boundary is a smooth surface S. Its displacement vector $v(q,t)$ satisfies the equation

$$\alpha \operatorname{grad} \operatorname{div} v - \beta \operatorname{rot} \operatorname{rot} v - \frac{\partial^2 v}{\partial t^2} = f,$$

in Q, where $\alpha=(\lambda+2\mu)/\rho$, $\beta=\mu/\rho$; assigned initial conditions on $v(q,0)$, $\partial v(q,0)/\partial t$; and boundary conditions of one of two kinds: $v(s,t)$ assigned on S, or

$$2\mu\frac{\partial v}{\partial n}+\lambda n \operatorname{div} v+\mu n\times \operatorname{rot} v+hv$$

assigned on S, where h is a given scalar function on S. The author constructs, in both cases, regular integral equations satisfied by the Laplace transform

$$u(q,\eta)=\int_\infty^0 e^{-\eta t}v(q,t)dt,$$

employing the method outlined in a previous paper [Dokl. Akad. Nauk SSSR (N.S.) **81** (1951), 513–516; MR **14**, 220]. *J. B. Diaz* (College Park, Md.).

See also: Zagustin, p. 828; Zurmühl, p. 828; Montroll, p. 836; Trostel, p. 850.

Fluid Mechanics, Acoustics

Batchelor, G. K. On steady laminar flow with closed streamlines at large Reynolds number. J. Fluid Mech. **1** (1956), 177–190.

The point made in this paper is that steady inviscid rotational flows with closed streamlines are indeterminate

unless a further condition is supplied. In the case of steady flow, this condition turns out to be an integral which must be satisfied by the vorticity distribution no matter how small the viscosity may be.

The inviscid flow equations are then combined with this integral condition in cases for which typical streamlines lie entirely in the region of small viscous forces. In two-dimensional closed flows, the vorticity is found to be uniform in a connected region of small viscous forces, with a value which remains to be determined by the condition that the viscous boundary layer surrounding this region must also be in steady motion. Analogous results are obtained for rotationally symmetric flows without azimuthal swirl, and for a certain class of flows with swirl having no interior boundary to the streamline, in an axial plane, the latter case requiring use of the fact that the vortex lines are also closed.

Y. H. Kuo (Peking).

Domm, U. Über die Wirbelstrassen von geringster Instabilität. Z. Angew. Math. Mech. **36** (1956), 367–371. (English, French and Russian summaries)

The stability of a vortex street is considered, using up to second degree terms in the differential equations for the perturbations. A one-parameter family of streets treated by Maue and Dolaptschiew is shown to be unstable, when the second degree terms are considered. The Kàrmàn street has the least instability in a certain sense. The notion of stability used here concerns the growth in time of the differences in the perturbations. This differs from the definition employed, for example, by the reviewer [J. Math. Phys. **30** (1952), 171–199; MR **13**, 790]. *E. A. Coddington* (Los Angeles, Calif.).

Ray, M. Two dimensional source or sink in a compressible fluid. Bull. Calcutta Math. Soc. **48** (1956), 69–74.

★ **Gerber, Sebastien. Étude théorique et expérimentale de la stabilité des chambres d'équilibre situées en aval d'une galerie en charge alimentée par un canal à écoulement libre.** Préface de L. Escande. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 320, 1956. v+122 pp. 1400 francs.

As introduction author reviews existing solutions for stability of surge tanks used in hydro-electric power stations. Relatively few configurations have been studied because of the non-linear friction terms in the equations of motion. Main part of paper is analysis of a restricted type of surge tank which is located on upstream side of the turbine and connected to the conduit. The inclusion of the open channel is the only original part of the author's analysis.

Equations of motion are set-up with considerable detail of physical phenomena and then reduced to dimensionless variables. Result is system of four differential equations which can be reduced for small oscillations to one second-order linear differential equation with non-constant coefficients. This equation is studied for stability under forcing function of unit step type. Criteria are derived for cases of infinitely long canal, finite canal, and restricted inlet to tank. In each case criteria are a minimum cross-section area to tank.

Last part of paper is a complete description of hydraulic model tests of the Saint-Cristaud plant in France. The criteria are applied to all results and are shown to be satisfied in all cases.

W. D. Baines (Ottawa, Ont.).

Greenspan, H. P. The generation of edge waves by moving pressure distributions. J. Fluid Mech. **1** (1956), 574–592.

At time $t=0$ a prescribed pressure distribution begins to move with constant velocity U parallel to a straight coast line bounding a beach of small constant slope, and sets up a wave motion which is studied here by means of linearized long-wave theory. When a Laplace transformation with respect to the time and a Fourier transformation with respect to the distance along the shore are applied, it is found that the transform of the wave elevation is the sum of an infinite number of discrete modes; a class of pressure distributions is chosen for which this sum reduces to its first term. By inversion of the transformations the elevation is expressed as a single contour integral, which is evaluated asymptotically when certain parameters (not precisely stated) are large. It is thus found that the front of the main wave motion travels with velocity U parallel to the shore while the rear boundary travels with velocity $\frac{1}{3}U$; this would also be expected from a group velocity argument. The wave period and duration found in this way agree with observations quoted by Munk, Snodgrass and Carrier [Science **123** (1956), 127–132]. In the last section the author considers another class of pressure distributions and concludes that only the fundamental edge-wave mode is effectively excited by such storms as are likely to occur in practice.

F. Ursell (Cambridge, Mass.).

Benjamin, T. Brooke. On the flow in channels when rigid obstacles are placed in the stream. J. Fluid Mech. **1** (1956), 227–248.

The various effects of lowering a rigid obstacle into a stream of ideal fluid flow under gravity in a horizontal channel are considered by applying the conservation laws of flow rate, energy and momentum. The conditions giving rise to different types of flow are examined; in particular, the conditions causing stationary waves in the downstream are distinguished from those causing a uniform supercritical receding stream. Precise calculations are carried out for the flow under a vertical sluice-gate and under an inclined plate. To account for the region near the solid obstacle, a conformal transformation method is used in which an unknown curve in the hodograph plane is approximated by an arc of an ellipse. The results compare favorably with solutions previously obtained by relaxation methods. A number of experiments are described. *T. Y. Wu* (Pasadena, Calif.).

Ertel, Hans. Eine Beziehung zwischen Phasengeschwindigkeit, Partikelgeschwindigkeit und Energie bei fortschreitenden permanenten Wellen. S.-B. Deutsch. Akad. Wiss. Berlin. Kl. Math. Phys. Tech. **1956**, no. 2, 12 pp.

From the classical hydrodynamical equations a relation is derived between particle velocity, energy density and phase velocity of waves. It is applied to the determination of the phase velocity of surface waves. *L. Van Hove.*

Jungclaus, G. Druckverteilungen bekannter Profiltypen bei inkompressibler Strömung. Z. Flugwiss. **5** (1957), 106–114.

Fogagnolo Massaglia, Bruna. Onde di Gerstner generalizzate. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955-56), 611–632.

Following Gerstner's method [see, e.g., Milne-Thomson,

Theoretical hydrodynamics, 3rd ed., Macmillan, New York, 1956, Ch. 14, § 80; MR 17, 796] the author seeks to satisfy the Lagrangian equations of motion of an inviscid liquid moving under gravity with a free surface of constant pressure, and constant undisturbed depth, by

$$x=\alpha+\varepsilon u(\beta,\tau),\quad y=\beta+\varepsilon v(\beta,\tau),$$

where u, v are periodic functions of the time τ. A method of successive approximation, here carried as far as ε^2, shows that (i) the wave-speed depends not only on wavelength but also on amplitude, (ii) the paths of the particles may be explained as ellipses with time-variable axes, (iii) except at the free surface the pressure depends on time. *L. M. Milne-Thomson* (Providence, R.I.).

Grohne, Diether. Über die laminare Strömung in einer kreiszylindrischen Dose mit rotierendem Deckel. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1956**, 263–282.

The steady laminar motion is calculated for water in a pillbox shaped vessel with circular cross-section when the top cover is rotating at a constant speed. The investigation is carried out in detail including a study of the flow in the corners. *C. C. Lin* (Cambridge, Mass.).

Morgan, A. J. A. On a class of laminar viscous flows within one or two bounding cones. Aero. Quart. **7** (1956), 225–239.

A special class of exact similarity solutions of the Navier-Stokes equations is presented in this paper for the incompressible, axially symmetric, laminar viscous flows within one or two bounding cones, the velocity component normal to the meridian planes being assumed to be zero. The solutions which are invariant under the one-parameter group of transformations (generalized uniform expansions) are such that the velocity components are inversely proportional to the distance from the cone vertex and their dependence on the polar angle is governed by ordinary differential equations which can be integrated exactly. Permeable conical boundaries are admissible for this class of flows with the suction or injection velocities inversely proportional to the distance from the vertex. These flow become identically quiescent when bounded by impermeable conical surfaces. *T. Y. Wu.*

Cooke, J. C. On Rayleigh's problem for a general cylinder. J. Phys. Soc. Japan **11** (1956), 1181–1184.

This is an improved version of the problem previously treated by Hasimoto [J. Phys. Soc. Japan **9** (1954), 611–619; **10** (1955), 397–405; MR **16**, 190, 967]. *Y. H. Kuo.*

Wadwa, Y. D. Slow viscous drag. Bull. Calcutta Math. Soc. **48** (1956), 45–46.

The author calculates the drag of an infinite cylinder with a lemniscate cross-section by a method originally due to Seth [Proc. Indian Acad. Sci. Sect. A. **40** (1954), 25–36; MR **16**, 82]. *Y. H. Kuo* (Peking).

Sanyal, Lakshmi. The equations of motion of an incompressible viscous fluid in the parallel system of coordinates. Indian J. Phys. **31** (1957), 6–10.

Batchelor, G. K. A proposal concerning laminar wakes behind bluff bodies at large Reynolds number. J. Fluid Mech. **1** (1956), 388–398.

This note advocates a model of the steady flow about a bluff body at large Reynolds number which is different from the classical free-streamline model of Helmholtz and Kirchhoff. It is suggested that, although the free-streamline model may be a proper solution of the Navier-Stokes equations with $\mu=0$, it is unlikely to be the limit, as $\mu\to0$, of the solution describing the steady flow due to the presence of a bluff body in an otherwise uniform stream. The limit solution proposed here is one which gives a closed wake.

A closed wake contains a standing eddy, or eddies, whose general features can be inferred from the results of an earlier investigation of steady flow in a closed region at large Reynolds number. In all cases, the drag (coefficient) on the body tends to zero as the Reynolds number tends to infinity. The procedure for finding the details of the closed wake behind two-dimensional and axisymmetrical bodies is described, although no particular case has yet been worked out. (Author's summary.)

Y. H. Kuo (Peking).

Ionescu, Dan Gh. Sur l'application de la théorie des fonctions de variable complexe dans l'hydrodynamique des fluides visqueux. I. Formules fondamentales. Com. Acad. R. P. Romîne **6** (1956), 981–984. (Romanian. Russian and French summaries)

In the present paper the author applies the Muskhelishvili-Vekua theory of singular integral equations to the problem of the slow motion of viscous incompressible fluids. The author transforms the fundamental equations into complex ones, and the different components of motion are obtained in terms of these new functions.

K. Bhagwandin (Blindern).

Ionescu, Dan Gh. Sur l'application de la théorie des fonctions de variable complexe dans l'hydrodynamique des fluides visqueux. II. Résolution du premier problème aux limites pour le cercle et la couronne circulaire, à l'aide des séries entières. Com. Acad. R. P. Romîne **6** (1956), 1059–1063. (Romanian. Russian and French summaries)

The necessary coefficients obtained in the previous paper [see preceding review] are determined explicitly. Exact expressions are obtained for the case of the rotational motion of the fluid between two co-axial cylinders, and the motion of a circular cylinder through the fluid bounded by a circular contour. *K. Bhagwandin.*

Czepa, Otto. Über das Filtergesetz der Grundwasserbewegung. Acta Hydrophys. **3** (1956), 181–192.

The author's object is to re-derive the fundamental filtration equations from the Navier-Stokes equations of motion for viscous incompressible fluids.

K. Bhagwandin (Oslo).

Kuwabara, Shinji. The forces experienced by two circular cylinders in a uniform flow at small Reynolds numbers. J. Phys. Soc. Japan **12** (1957), 291–299.

Using Imai's general method of solving two-dimensional Oseen's equations, the drag and lift coefficients for one of two parallel circular cylinders in a uniform flow at small Reynolds numbers have been calculated. The general expressions for these coefficients become very simple when the two cylinders are placed symmetrically with respect to the uniform flow. Numerical calculations have been carried out in several cases for various values of the parameters. *Author's summary.*

Hasimoto, Hidenori. Boundary layer growth on a flat plate with suction or injection. J. Phys. Soc. Japan 12 (1967), 68–72.

The author treats the problem of an unsteady flow produced by a porous plate started impulsively with (1) a velocity varying arbitrarily with time and a uniform suction; and (2) a velocity increasing with some power of time and a suction velocity being inversely proportional to the square root of time. For both cases, solutions satisfying the Navier-Stoke equations are obtained.

Y. H. Kuo (Peking).

Cheng, Sin-I. Some aspects of unsteady laminar boundary layer flows. Quart. Appl. Math. **14** (1957), 337–352.

The author considers the problem of the motion of a semi-infinite plate parallel to itself in a viscous fluid, starting from rest. The issue examined is the problem of joining the solution at large distances with the solution at small distances. The proposed method is to develop power series solutions in one of the variables from an intermediate station. It seems that this method may lead to satisfactory numerical solutions, but still leaves much to be desired from the mathematical point of view.

C. C. Lin (Cambridge, Mass.).

Görtler, Henry. A new series for the calculation of steady laminar boundary layer flows. J. Math. Mech. **6** (1957), 1–66.

A general method is developed for solving problems of plane and steady laminar boundary layer flows in incompressible fluids with arbitrary outer pressure distribution. This method is based on the introduction of the dimensionless quantities

$$\xi=\frac{1}{\nu}\int_0^x U(x)dx,\qquad \eta=U(x)y\left\{2\nu\int_0^x U(x)dx\right\}^{-\frac{1}{2}}$$

where the customary notation is used. The method has a number of desirable features. Some of these are the following: (1) The leading term of the new series satisfies exactly the outer boundary condition at all cross-sections along the wall. This makes the initial approximation very accurate. (2) The method applies to arbitrary conditions, blunt or sharp, at the leading edge. (3) With the help of tabulated universal functions, the method can be easily applied in any given case. Several examples are worked out, some involving separation. To predict separation in the difficult case of the Schubauer ellipse, however, still requires special treatment.

C. C. Lin.

★**Batchelor, G. K.; and Townsend, A. A. Turbulent diffusion.** Surveys in mechanics, pp. 352–399. Cambridge, at the University Press, 1956. $9.50.

This paper gives an up-to-date account of the basic knowledge of turbulent diffusion by setting down the essential ideas as distinct from empirical rules used in the analysis of problems. As is well-known in this difficult subject of turbulent diffusion, most of the theoretical developments merely help to provide a framework for the interpretation of measurements, and do not provide definite deductions. The authors, however, have managed to collect a number of cases where a definite contact has been established between theory and experiment. They also cautioned against undue optimism by bringing up cases where this agreement is only partially successful. For example, it is pointed out that although Batchelor's prediction of the relative separation between two particles is in apparent agreement with Richardson's observations, the cases considered in the theory and experiment are really for different physical circumstances. The reader will also find an illuminating account of the theory of the deformation of material lines and surfaces in turbulent diffusion where definite deductions are obtained and yet the state of our knowledge is not entirely satisfactory. Throughout the paper, the importance of molecular diffusion (including heat conduction and viscosity) is emphasized.

C. C. Lin (Cambridge, Mass.).

Batchelor, G. K., and Proudman, I. The large-scale structure of homogeneous turbulence. Philos. Trans. Roy. Soc. London. Ser. A. **248** (1956), 369–405.

The authors show that the influence of pressure forces will in general invalidate the older conclusion of the permanence of large eddies. In the general case of anisotropic turbulence, the velocity covariance $\overline{u_iu_j'}$ is found to be of order r^{-5} for large distances of separation, and thus the Loitsiansky integral is divergent. In the special case of isotropic turbulence, the authors conclude that $\overline{u_iu_j'}$ is no larger than $O(r^{-6})$, making the Loitsiansky integral convergent. However, the constancy of the integral no longer holds, because of the behaviour of the triple correlation function for large distances of separation. In the final period of decay, the decay of energy follows the same law as before, but the correlation function no longer has a simple form.

C. C. Lin.

Velikanov, M. A. Coarse grained turbulence and the formation of river beds. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 71–78. (Russian)

Yasuhara, Michiru. On the hypersonic viscous flow past a flat plate with suction or injection. J. Phys. Soc. Japan **12** (1957), 177–182.

The author calculates the flow between the shock and a porous flat plate with a suction or injection velocity proportional to the pressure in the case of large Mach number. Following the method of Stewartson [Proc. Cambridge Philos. Soc. **51** (1955), 202–219; J. Aero. Sci. **22** (1955), 303–309; MR **16**, 638, 877] the effects of suction and injection on the character of the flow are investigated. It is concluded that injection has a larger influence than that of suction.

Y. H. Kuo (Peking).

Heaslet, Max. A.; and Fuller, Franklyn B. Particular solutions for flows at Mach Number 1. NACA Tech. Note no. **3868** (1956), 32 pp.

Six one-parametric solutions of the equation for transonic plane or axially symmetric flows are determined explicitly in the physical plane by separation of variables. Two solutions are subsonic plane, two are supersonic plane, one is subsonic axial and one is supersonic axial. In a special case one of the supersonic plane solutions corresponds to a Prandtl-Meyer wave. In both the plane and axial cases a particular supersonic flow can be connected to a subsonic flow. The resulting flows are past a one-parametric family of walls with discontinuous slope. Pressure is determined by numerical integration.

C. S. Morawetz (New York, N.Y.).

Murgulescu, Elena. Sur le mouvement conique dans le cas d'une aile extérieure au cône caractéristique du sommet. Com. Acad. R. P. Romîne **6** (1956), 1179–1185. (Romanian. Russian and French summaries)

The present work is a continuation of the author's previous results concerning conical flow [same Com. **2**

(1952), 489–494; MR 17, 551]. The velocity components are obtained in the form of definite integrals. These integrals are not discussed explicitly, however.
K. Bhagwandin (Blindern).

Oswatitsch, K. Die Berechnung wirbelfreier achsensymmetrischer Überschallfelder. Österreich Ing.-Arch. 10 (1956), 359–382.

Treating first the case of axisymmetric flow at Mach numbers just above one, the author develops a characteristics procedure that is almost as simple to use as the linearized characteristics method of Sauer and Heinz [Oswatitsch, Gasdynamik, Springer, Vienna, 1952; MR 14, 814]. To accomplish this, the variables $(2/3)u^{3/2}$ and vy are employed. (u, v are dimensionless streamwise and radial velocity perturbations, and y the radial coordinate.) After working this method out in detail and showing examples of its application in transonic flow past bodies of revolution, the author proceeds to general irrotational supersonic axisymmetric flow. Accuracy and simplicity are achieved by use of u and ρyv as variables, where ρ is the fluid density. This procedure is worked out, including a diagram for determination of Mach-line slope, and examples are exhibited. In these the exact construction is compared with the Sauer-Heinz, and in one case (1:6 spindle at five Mach numbers) the calculated pressure distributions are compared with experimental results.
W. R. Sears (Ithaca, N.Y.).

Etkin, B. Aerodynamic transfer functions: an improvement on stability derivatives for unsteady flight. University of Toronto, Institute of Aerophysics, Rep. no. 42 (1956), iii+11 pp. (3 plates).

The author shows how, by means of Laplace transforms, it is possible to obtain certain quantities called aerodynamic transfer functions. The usual stability derivatives of aerodynamic theory are simply related to, and are in fact, an approximation to these transfer functions. The use of transfer functions is valuable in problems for which stability derivatives are inadequate, e.g. when rapid changes of forces are involved as in the penetration of gusts.
G. N. Lance (Southampton).

Legendre, Robert. Aile conique à bords d'attaque subsoniques. C. R. Acad. Sci. Paris 244 (1957), 1878–1880.

L'auteur fournit les expressions générales des composantes des vitesses de perturbation pour un écoulement conique linéarisé autour d'un obstacle applati, sans distinguer le cas symétrique du cas portant et pour des donneés générales. Ces expressions font évidemment intervenir la fonction de Green du domaine sur lequel est fait la représentation de l'écoulement, qui est ici calculée à l'aide des fonctions elliptiques. L'auteur applique ces résultats pour obtenir la condition à laquelle doivent satisfaire les pentes pour que la pression reste finie au bord d'attaque.
P. Germain (Paris).

Stetter, Hans J. Beiträge zum Wechselwirkungsproblem in linearisierter Überschallströmung. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. 1956, 61–86 (1957).

Ce travail est consacré à l'étude de l'écoulement supersonique linéarisé autour d'une combinaison aile fuselage. Essentiellement, l'auteur expose certaines améliorations qui peuvent être apportées dans l'emploi de la méthode proposée par J. N. Nielsen [Thesis, Cal. Inst. Tech. 1951] et reprise dans Nielson and Pitts, NACA Tech. Note no. 2677 (1952). La théorie de ces écoulements est présentée en faisant appel à la théorie des distributions, ce qui fournis une exposition simple et élégante. Certaines difficultés provenant de développements en série de Fourier de fonctions discontinue sont surmontées. Des procédés systématiques sont donnés pour le calcul de certains coefficients, procédés particulièrement faciles à mettre en oeuvre avec une machine à calculer moderne. Signalons enfin que cet article est le résumé de la thèse de l'auteur présentée en juillet 1955 à la faculté des sciences de la "Technische Hochschule" de Münich.
P. Germain.

Drummond, W. E. Interaction of nonuniform shock waves. J. Appl. Phys. 28 (1957), 76–85.

Une onde de choc non uniforme rencontre la surface de séparation de deux milieux; quel écoulement va-t-on observer? Tel est l'objet de cet article. Par hypothèse les chocs sont supposés faibles, et les variations d'entropie sont négligées; par choc non uniforme, l'auteur entend onde de choc suivie d'un régime d'ondes simples. Le problème est traité aussi bien dans le cas d'écoulements unidimensionnels non stationnaires que dans le cas des écoulements bidimensionnels stationnaires. On observe dans tous les cas un choc réfracté, ou transmis, et selon les propriétés des deux milieux, un choc ou une onde simple réfléchi. La détermination complète de l'écoulement est effectuée par la méthode numérique basée sur la théorie des caractéristiques.
P. Germain (Paris).

Jahn, Robert G. Transition processes in shock wave interactions. J. Fluid Mech. 2 (1957), 33–48 (4 plates).

The author reviews the experimental work in shock refraction and reflection and points out similarities in the problems involved in explaining the results. In particular he discusses the persistence of regular reflection and the inability of three-shock theory to explain the observed Mach configurations for weak incident shocks. A mechanism for "explaining" the experimental results is proposed; namely, a singularity at the intersection of the shocks is postulated which is of the same nature as the singularity that exists in subsonic flow over sharp corners. This singularity presumably exists in the reflection and refraction problems in the angular region between the limit of regular reflection (refraction) and the onset of Mach reflection (refraction) and in the case of Mach reflection for weak incident shocks.
A. H. Taub.

★ **Donovan, A. F.; and Lawrence, H. R. Aerodynamic components of aircraft at high speeds.** Princeton University Press, Princeton, N. J., 1957. xiv+845 pp. $17.50.

"The Princeton series" continues on its majestic way, compendious, unexciting, indispensable.

The present volume, "High speed aerodynamics and jet production, vol. 7," is one of a series of three (vols. 6, 7 and 8) on external aerodynamics. It aims at the systematic presentation of results on the aerodynamics of wings (Section A, 278 pages, by R. T. Jones and Doris Cohen, together with Section F, on unsteady problems, 136 pages, by I. E. Garrick), of bodies (Section B, 37 pages, by C. E. Brown), of propellers (Section D, 34 pages, by C. B. Smith), of wing-body-tailplane-propeller combinations (Section C, 271 pages, by C. Ferrari) and of diffusers and nozzles (Section E, 72 pages, by J. C. Evvard). Experimental methods are not treated, being postponed, together with the study of the aerodynamics of the aeroplane as a whole, to vol. 8. However, there is a short

section on experimental results (Section G, 39 pages, by C. W. Frick).

The main theories fundamental to external aerodynamics were given in fuller detail in vol. 6. In addition, vols. 3, 4 and 5 contain much material relevant to the subject. Obviously, therefore, there has been no deliberate avoidance of overlapping between different volumes, and the aim has been to include everything rather than to avoid duplication or even triplication of treatment. Thus, when Professor Ferrari states on p. 540 of vol. 7 that no further exposition of the Evvard type of approach to supersonic wing theory is necessary, because it has been sufficiently explained in vol. 6, and then proceeds to give four further pages of such exposition, the reader may well remember also the full discussion on pages 177–190 of vol. 7 itself. This lavish approach at least means that a reader can take his choice between different directions of approach to leading theories.

There is some danger, however, that authors of different sections of the series may have adopted approaches so similar that certain kinds of treatment which one would expect to find in a series of this size are absent. When Dr. Max Munk criticized vol. 6 for not presenting the results of aerodynamic research in a form suitable for direct use in design, the authors of that volume, including this reviewer, felt that we could answer by pointing to the future vol. 7. Now that it has appeared, it is difficult to be so sure. An academic reviewer can feel nothing but admiration for the lucid treatment of the basic theory of the steady aerodynamics of wings which Mr. and Mrs. Jones have given in section A, as well as for the similar types of treatment in sections B, D and F, but it cannot be pretended that these constitute substantially more than brilliant expositions of theory. Section A contains several comparisons with experiment in the single field of lift slope at zero lift (figs. 3c, 4h, 7p, 7t, 7u, 14a, 14c, 14i, 14l, 14s and 14t) but hardly any in other fields, including the crucial ones of wave drag at zero lift and maximum lift; and the behaviour of wings at high angles of attack, which is so important to aircraft designers, is barely touched on. Again, in section D, we have only a theoretical analysis of the possibilities of the supersonic airscrew, with three references to literature and no experimental results. In section F, the author suggests that theory is at a definite advantage over experiment in unsteady wing aerodynamics, and accordingly includes only one set of experimental results; but his contention would be more convincing if the existing theories could treat the many aerodynamic oscillations in which shock wave-boundary layer interactions have been shown to play an essential part.

In the brief space available to him, the author of section G does something to restore the balance, and it must be made clear that these criticisms do not apply to the thoroughly practical section E on diffusers and nozzles. The momental section C on interaction problems also does something not attempted elsewhere in the series, or indeed in the whole aerodynamical literature. The effects of the trailing vortices and wake of a wing on the tailplane, of the propeller slipstream on the wing, of the wing on the propeller (which, by the way, is of the ordinary subsonic kind, not that treated in section D), and of the wing and body in combination, are treated for subsonic flow in 133 pages, and followed by 45 pages on the trailing vortex systems and 91 pages on wing-body interactions in supersonic flow. The theories are presented in a very practical form, and the number of experimental comparisons is substantial, considering the complexity of the subject. (Only the use of English words is not always quite accurate, a point for which the author cannot reasonably be blamed.)

To sum up ,the whole book could hardly be bettered as a primer in high-speed aircraft aerodynamics for members of the enormous band of theoretical aerodynamicists which the aircraft- and missile-building countries seem now to require, but only parts of it will be more than useful background reading to more practical men.

The index, unfortunately, is so incomplete as to be useless. To illustrate this, consider the references to G. N. Ward and his theories in this book. Substantial accounts are given of (i) the Ursell-Ward flow-reversal theory (pp. 62-4), (ii) the Ward slender-body theory (pp. 246–253), (iii) the Ward quasi-cylinder theory (pp. 272–274), (iv) the Ward wing-body interaction theory in the subsonic case (pp. 343–360), and (v) the wing theory discovered independently by Evvard, Ward and Krasilshchikova and called the Evvard-Krasilshchikova theory in pp. 177–190 but the Evvard-Ward theory in pp. 540–543. Now, the index omits G. N. Ward altogether, though referring under K. E. Ward to pp. 335f (where papers by Jacobs and K. E. Ward are discussed), p. 333 (where no Ward is mentioned at all) and p. 540 (where the Evvard-Ward theory is mentioned). There is no reference to (i) even under Ursell or flow reversal, or to the main account (ii) of slender-body theory under that or any other name, or to (iii), though (iv) is referred to under "slender wing-body combinations". The index is long, certainly, but only as a result of the unimaginative multiplication of single-reference items, such as "vertical location of legs of horseshoe vortex", which no reader would think of looking up. The value of a compendium of this kind, otherwise so great, is seriously reduced by an inadequate index, and the Table of Contents is not precise enough to make up for the deficiency. *M. J. Lighthill.*

Vincenti, Walter G.; Wagoner, Cleo B.; and Fisher, Newman H., Jr. Calculations of the flow over an inclined flat plate at free-stream Mach number 1. NACA Tech. Note no. **3723** (1956), 70 pp.

The solution for compressible inviscid flow past an inclined plate with free stream Mach number 1 is found by a) solving a finite difference scheme in the hodograph plane for the flow below the plate and b) using a standard form of the method of characteristics in the purely supersonic flow above the plate. The body of paper consists in a). The stream function ψ satisfies an equation of Tricomi type in a domain bounded by two coordinate lines in the elliptic region and four characteristics in the hyperbolic region. ψ vanishes on the elliptic boundary and two characteristics and has a prescribed singularity on the parabolic line. The singularity is treated by an unconventional expansion procedure. The difference scheme requires eight types of difference operators. The method of iteration is the following: a) Prescribe the values of what is equivalent to a certain oblique derivative on the sonic line, b) find the solution in the elliptic region and on the sonic line, and c) solve the hyperbolic problem where ψ is prescribed on the characteristic and the sonic line. This yields new values of the oblique derivative of ψ on the sonic line. Convergence of the iteration or the difference scheme is not proved. Transformation to the physical plane involves no difficulties. Results for a plate at an inclination of 13° are given. They illustrate in contrast to the incompressible case that the stagnation point is

extremely close to the front end of the plate and the disturbance is in general confined to a very small neighborhood. The results of Guderley [J. Aero. Sci. 21 (1954), 261–274] are confirmed. *C. S. Morawetz.*

Guderley, Gottfried. On transonic airfoil theory. J. Aero. Sci. 23 (1956), 961–969.

Transonic flow past a double wedge airfoil of large aspect ratio, $1/t$, is treated as a small deviation from sonic parallel flow. The flow is studied by solving approximately for the Legendre potential ϕ in the (η, θ, y)-space, where η is a function of speed in the x, z-plane and θ is flow angle in the same plane. ϕ satisfies a mixed equation where the derivatives with respect to y occur only in nonlinear terms:

$$\phi_{\eta\eta}-\eta\phi_{\theta\theta}= \phi_{yy}(\phi_{\eta\eta}\phi_{\theta\theta}-\phi_{\eta\theta}^2)-\phi_{\eta y}^2\phi_{\theta\theta}-\phi_{\theta y}^2\phi_{\eta\eta}+2\phi_{\eta\theta}\phi_{\eta y}\phi_{\theta y}.$$

To lowest order the flow in any cross-section, y=const, is a solution of $\phi_{\eta\eta}-\eta\phi_{\theta\theta}=0$ with y and t occurring as parameters. This solution is found to be

$$t^{0.6}d^0(y)+tc(y)\phi_{-1}(\eta, \theta)+I,$$

where ϕ_{-1} is a solution of the Tricomi equation with an appropriate singularity. I is a solution which is different in two subregions of the flow. The functions c and d depend on shape. Near the profile for small t, I behaves like $t^{2.2}$ as $t\to 0$. The corresponding pressure terms behave like $t^{1.2}$. The dominant parts of the correction to this approximation behave like $t^{1.6}$. Correction: p. 963 line 12 from bottom, read "cannot" for "can".

C. S. Morawetz (New York, N.Y.).

Sanders, Karl L. The optimum design of long-range aircraft. An examination of the best values of aspect ratio wing loading and fuel load ratio for different conditions. Aircraft Engrg. 29 (1957), 98–106.

Patraulea, N. N. Le mouvement à symétrie axiale autour des ailes annulaires minces. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 8 (1956), 115–119. (Romanian. Russian and French summaries)

The author develops a vortex theory, similar to that of the motion around plane thin profiles, for the axial symmetrical flow around a thin annular wing. The solution is expressed in the form of a singular integral equation. An approximate solution is presented. This solution is however not uniquely determined. *K. Bhagwandin.*

Albring, Werner. Ein Näherungsverfahren zur Konstruktion ebener stossfrei angeströmter Schaufelgitter und Berechnung ihrer Druckverteilung bei stationärer Strömung. Wiss. Z. Tech. Hochsch. Dresden 5 (1955/56), 65–78.

Through the use of meshes (flow nets) of streamlines and equipotential lines, the author establishes a geometrical procedure which yields a simple and rapid construction of a two-dimensional unstaggered cascade with prescribed design characteristics and which permits a rapid computation of the irrotational flow of an incompressible or compressible fluid through the cascade. The actual construction consists in fitting together discs of various sizes (circular for the hydrodynamic case, and elliptical for the aerodynamic case) to determine the streamlines and equipotential lines. The circle construction for the hydrodynamic case has been previously developed and extensively used by B. Eck and S. Leliavsky. Some specific examples are worked out in detail. As a check of the effectiveness of the method, the lift coefficient obtained from the calculated pressure distribution is compared with that obtained from momentum considerations. The greatest difference found is one of 5% in a compressible flowcalculation. *P. Chiarulli.*

Van Dyke, M. D. The slender elliptic cone as a model for non-linear supersonic flow theory. J. Fluid Mech. 1 (1956), 1–15.

In this paper the second order slender body theory for an unyawed elliptic cone is derived and used to study the accuracy of various less exact approximations. Assuming the flow and the cone-axis to be in the z-direction, and neglecting terms whose effect will be of order higher than second, the flow is governed by the equation potential φ,

$$(*)\quad \varphi_{xx}+\varphi_{yy}=\beta^2\varphi_{zz}+2M^2(\varphi_x\varphi_{xz}+\varphi_y\varphi_{yz})+ (\gamma+1)M^4\varphi_z\varphi_{zz}+M^2(\varphi_x^2\varphi_{xx}+2\varphi_x\varphi_y\varphi_{xy}+\varphi_y^2\varphi_{yy}),$$

together with appropriate boundary conditions; here $\beta^2=M^2-1$, M is the free stream Mach number. A first approximation ("first order slender body theory") is obtained by solving (*) with right-hand side set equal to zero. The desired second order solution (called "exact" in the paper) is then gotten by substituting the first order solution in the right hand side of (*) and solving. (Throughout, calculations are performed in elliptic conical coordinates.) Formulas for pressure and drag coefficients are given. For "flat" elliptic cones a number of approximate theories are compared graphically with the "exact" theory and with experimental results. For such cones it is noted, among other things, that the effects of the triple products in (*) may be neglected except in a small region near the leading edge, where their contribution is significant; and that the linear theory (i.e., neglecting non linear terms in (*)) fails to predict the leading edge drag correctly. The paper concludes with a discussion of the accuracy of Ferri's "linearized characteristics method" [NACA Rep. no. 1102 (1952); MR 15, 367]. *W. Littman* (Berkeley, Calif.).

Tipei, N.; et Constantinescu, V. N. La lubrification des surfaces sphériques. Com. Acad. R. P. Romîne 6 (1956), 531–536. (Romanian. Russian and French summaries)

The authors derive solutions of the lubrication equation for the region bounded by two spherical surfaces. Expressions are obtained for particular and general boundary conditions. These solutions are nevertheless of an approximate nature; no rigorous proofs are presented.

K. Bhagwandin (Blindern).

See also: Halilov, p. 827; Nielsen, p. 829; Miles, p. 839; Ursell, p. 848; Agostinelli, p. 849; Hamaguchi, p. 856; Karpman, p. 856; Blinova, p. 858; Gutman, p. 858.

Optics, Electromagnetic Theory, Circuits

Lacomme, Pierre. Influence du chromatisme sur les images de diffraction. Rev. Opt. 36 (1957), 1–19, 71–87.

The author discusses first the well-known formulae for correcting an objective for three colors in the Gaussian region for systems of thin lenses and investigates the deviations due to lens thickness. He continues investi-

gating the Seidel aberrations as a function of wavelength, giving rules for achromatising for artificial and for daylight illumination. He also considers the diffraction image as a function of wavelength. *M. Herzberger.*

MacAdam, D. L. Analytical approximations for color metric coefficients. J. Opt. Soc. Amer. **47** (1957), 268–274.

The author discusses a suggestion by von Schelling, assuming that the color space is non-euclidean, but of constant negative curvature, and that the surfaces of subjectively equal brightness are its geodesic surfaces. Since the results seemed to be in agreement with experimental results on color matching, the author tried to apply them to recent data on color discrimination. Trying to obtain the best fit of the data with a least square approximation led, however, to results which deviate far too much from the experimental values obtained. The author believes that a space of constant curvature cannot fit color discrimination space which seems to have positive curvature near the white point and approaches negative curvatures in the neighborhood of pure spectral colors. *M. Herzberger* (Rochester, N.Y.).

Durand, Emile. Les équations de l'électromagnétisme non conservatif déduites d'une intégrale d'action invariante. J. Phys. Radium (8) **17** (1956), 1016.

A generalized form of Maxwell's equations applicable to the case where electric charges are created or annihilated was previously derived by the author [C. R. Acad. Sci. Paris **242** (1956), 1862–1865; MR **7**, 1257]. It is shown here that these equations can also be derived from the condition that the variation of a generalized Lagrangian should vanish. The case where fictitious magnetic charges are created is also considered. *J. E. Rosenthal.*

Kiyono, Takeshi. Über die stationären und quasistationären magnetischen Felder des Rotationsellipsoides. Mem. Fac. Engrg. Kyoto Univ. **18** (1956), 209–236.

This paper contains a number of calculations on the magnetic fields near a conducting ellipsoid of rotation: (a) placed in a stationary current field; or (b) placed in a quasistationary a.c. field. *P. W. Anderson.*

Braginskiĭ, S. I. On the theory of motion of charged particles in a strong magnetic field. Ukrain. Mat. Ž. **8** (1956), 119–126. (Russian)

Let a charged particle of mass m and charge e be moving in a strong magnetic field $\mathbf{H}$ under the influence of a relatively weak electric field $\mathbf{E}$. Let $\boldsymbol{\omega}=e\mathbf{H}/mc$, let the position vector of the particle be $\mathbf{r}$ and let the velocity be $\mathbf{v}$. Let $\omega=|\boldsymbol{\omega}|$. Let $v_{\parallel}$ and $v_{\perp}$ denote the scalar components of $\mathbf{v}$ which are respectively parallel to $\mathbf{H}$ and perpendicular to $\mathbf{H}$. The author derives approximate expressions for $d\mathbf{r}/dt$, $dv_{\parallel}/dt$, and $dv_{\perp}/dt$ by using an asymptotic expansion, valid for large values of ω, and neglecting terms involving powers of ω^{-1} higher than the first. This article may be viewed as a sequel to a previous article by Bogolyubov and Zubarev [same Ž. **7** (1955), 5–17; MR **17**, 217], and employs certain equations thereof.

Certain simplifying approximations are needed for the applicability of the results obtained here. Let L be, roughly speaking, the smallest distance in which $\mathbf{E}$ and $\mathbf{H}$ change appreciably, and let t be the smallest time over which the magnetic field changes appreciably. Then we must assume: $v/L\ll\omega$, $E\ll(v_{\perp}/c)H$ (so that the particle can never come to rest), and $1/t\ll v/L$. Moreover, the mean velocity of the center of the Larmor circle must be small as well as the rate of change of the components of this mean velocity. Under the above assumptions the results include, among others:

$$(1)\quad d\mathbf{r}/dt=v_{\parallel}\boldsymbol{\epsilon}_0+(1/\omega)[\mathbf{F}\boldsymbol{\epsilon}_0]-(v_{\parallel}^2/\omega)[(\boldsymbol{\epsilon}_0\nabla)\boldsymbol{\epsilon}_0,\boldsymbol{\epsilon}_0]-(v_{\perp}^2/2\omega)[(\nabla\omega/\omega),\boldsymbol{\epsilon}_0],$$

$$(2)\quad (d(v_{\parallel}^2+v_{\perp}^2)/dt)/2=\mathbf{F}(d\mathbf{r}/dt)+(v_{\perp}^2/2\omega)(\partial\omega/\partial t),$$

$$(3)\quad d(v_{\perp}^2/\omega)/dt=0,$$

where ∇ is the gradient operator, [ ] denotes cross-product of vectors, two vectors written side by side denotes inner product, and $\boldsymbol{\epsilon}_0$, $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ is a variable orthonormal frame with $\boldsymbol{\epsilon}_0\parallel\mathbf{H}$. In (1), the first term on the right side represents the motion of the particle along a magnetic line of force, the second is a "drifting" across the magnetic field due to the component of $\mathbf{E}$ perpendicular to $\mathbf{H}$, the third term represents a "centrifugal drifting" due to the curvature of the magnetic lines of force, while the last term is a "magnetic drift" due to the inhomogeneity of the magnetic field. *W. L. Baily.*

Tonolo, Angelo. Sulla determinazione del campo elettromagnetico all'interno di un conduttore omogeneo e isotropo. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 403–408.

The problem solved in this paper is the determination of the electric and magnetic forces in a homogeneous isotropic conducting medium at any point within a given fixed closed surface, given the forces initially everywhere within the surface and subsequently everywhere on it. *E. T. Copson* (Saint Andrews).

Backus, George. The external electric field of a rotating magnet. Astrophys. J. **123** (1956), 508–512.

H. Alfven in his book "Cosmical electrodynamics" [Oxford, 1950; MR **12**, 756] has given the formula (1) $\mathbf{E}=-(\boldsymbol{\omega}\times\mathbf{r})\times\mathbf{B}$ for the electric field at position $\mathbf{r}$ due to a magnet with field $\mathbf{B}$ rotating with angular velocity $\boldsymbol{\omega}$. The author questions this formula. He computes what he considers to be the correct formula for $\mathbf{E}$ in various particular cases directly from the Maxwell field equations, deriving results inconsistent with (1). The results of his method are shown to agree with the experimental evidence cited by Alfven in favor of his formula. *O. Frink.*

Bolinder, E. Folke. The classical electromagnetic equations expressed as complex four-dimensional quantities. J. Franklin Inst. **263** (1957), 213–224.

If the four-dimensional relativistic quantities include complex numbers, the classical electromagnetic equations can be compressed into three terms with three relations between them. (The bibliography does not include references to analogous relations in the literature of quantum electrodynamics.) *G. Kron* (Schenectady, N.Y.).

Hofmann, H. Über den Kraftangriff des Magnetfeldes an Elementarströmen. Österreich. Ing.-Arch. **11** (1957), 1–5.

Picht, Johannes. Zur Theorie der Totalreflexion. Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Phys. Tech. **1955**, no. 2, 54 pp. (1956).

Die wellenoptische Theorie der Totalreflexion ist bekannterweise deshalb eine sehr interessante Frage in der Theorie der Ausbreitung von elektromagnetischen Wellen, weil nach der — entgegen dem Resultate der geo-

metrischen Optik — die Energie auch in das optisch dünnere Medium eindringt, dasselbe jedoch wieder verlässt. Ganz analoge Probleme treten in der wellenmechanischen Theorie des Durchganges eines geladenen Teilchens durch einen Gamowschen Potentialberg auf und neuerdings hat man interessante Totalreflexionserscheinungen bei der Ausbreitung von Erdbebenwellen gefunden. Die aus der Lehrbuchliteratur bekannte Theorie der Totalreflexion, die grösstenteils auf den Arbeiten von A. Eichenwald beruht, betrachtet ein paralleles einfallendes Strahlenbündel. Im optisch dünneren Medium nehmen dann die Feldstärken exponentiell ab. In der vorliegenden Arbeit beschäftigt sich der Verfasser — nach einer Zusammenfassung seiner Resultate bezüglich der Integraldarstellungen beliebig deformierter Wellen — mit dem Problem, dass ein Hertzscher Dipol erstens senkrecht und zweitens parallel zur Trennungsebene der zwei Medien orientiert ist. Die vom schwingenden Dipol ausgehende Welle (welche der Verfasser eine Kugelwelle nennt) zerlegt er dann in ebene Teilwellen, auf die man einzeln die erwähnte Theorie anwenden kann. Das Endresultat erhält man durch eine Integration über alle diese Teilwellen. Das Ergebnis kann als eine Superposition einer vom idealen Bildpunkt ausgehenden Kugelwelle und von einer Zusatzwelle erster bzw. zweiter Art (die jedoch keine Kugelwellen sind) dargestellt werden. Weiter ergibt sich, dass diese Zusatzwellen eine (scheinbare) Eindringtiefe und eine "sphärische Aberration" besitzen. Betrachtet man die Resultierende der reflektierten Kugelwelle und der Zusatzwelle, so besitzt auch die eine scheinbare Eindringtiefe und eine sphärische Aberration. Im Falle des zur Trennungsfläche parallelen Dipols tritt in der reflektierten Welle ausserdem eine Komponente auf, deren Hertzscher Vektor senkrecht auf die Richtung des Dipols steht, also scheinbar von einem so orientierten schwingenden Dipol herrührt. Ausserdem folgt, dass im Falle des zur Trennungsebene parallelen Dipols auch noch ein Astigmatismus der reflektierten Welle entsteht. Das Problem, dass nach einer sehr dünnen Schicht des optisch dünneren Mediums wieder eine dichtere folgt, wird in der vorliegenden Arbeit nicht besprochen.

T. Neugebauer (Budapest).

Ursell, F. On the short-wave asymptotic theory of the wave equation $(\nabla^2+k^2)\phi=0$. Proc. Cambridge Philos. Soc. **53** (1957), 115–133.

A closed convex curve in two dimensions, satisfying certain regularity conditions, is emitting short waves towards infinity, the normal velocity amplitude $V(s)$ being prescribed on the curve as a function of the arc length s. The potential $\phi(s)$ on the curve satisfies all the integral equations

$$\phi(s')-\tfrac{1}{2}i\int \phi(s)\frac{\partial G}{\partial n}(s, s')ds=-\tfrac{1}{2}i\int V(s)G(s, s')ds,$$

where $G(s, s')$ is any Green's function of the problem. An asymptotic and convergent short-wave solution can be found by iteration if G can be chosen explicitly so that the integral equation has a kernel that is small in the sense defined by the author. At any point of the curve draw the local circle of curvature; then the explicit known solution (derived in the appendix) for a source on this circle is, with slight modification, a possible G adequate for the purpose in mind. The leading term in the resulting asymptotic expansion of $\phi(s)$ is $-ik^{-1}V(s)$, if $V(s)\neq 0$, in which k is the (large) wave number. "The present work appears to be the first practical and rigorous solution of a short-wave problem in optics or acoustics when a solution in closed form is not available".

C. J. Bouwkamp (Eindhoven).

Nomura, Yûkichi; and Katsura, Shigetoshi. Diffraction of electromagnetic waves by ribbon and slit. I. J. Phys. Soc. Japan **12** (1957), 190–200.

The authors solve the problem of the diffraction of an electromagnetic wave by a ribbon or slit on the assumption that the plane of incidence is perpendicular to the edge. The diffracted field is derived from electric and magnetic Hertzian vectors, both being parallel to the edge, and expanded in terms of Weber-Schafheitlin integrals and hypergeometric polynomials. Numerical results for quantities like scattering and transmission coefficients will be given in a subsequent paper.

C. J. Boukamp.

★ **Copson, E. T. Some applications of Riesz's method.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 107–113. University of Maryland Book Store, College Park, Md., 1956.

Marcel Riesz's method of solving Cauchy's problem for the wave equation is shown to be applicable to two simple problems for which the method was not originally devised, viz. (1) that of diffraction by a plane screen, (2) that of Cauchy for damped waves.

C. J. Bouwkamp.

Epstein, Irving J. On a Fredholm equation in diffraction theory. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-20** (1956), i+31 pp.

The integral equation referred to in the title is that of Magnus [same Rep. No. **EM-80** (1955); MR **17**], 165 , viz.

$$\frac{-i\alpha y}{\sqrt{\pi}}=E(y)+\frac{2\alpha}{\pi}\int_0^1 G(x, y; \alpha)E(x)dx\ (0\leqq y\leqq 1),$$

in which the kernel is

$$G(x, y; \alpha)=-\frac{\pi}{4}\int_{|x-y|}^{x+y}\frac{J_1(\alpha\tau)+iH_1(\alpha\tau)}{\tau}d\tau,$$

where J_1 is a Bessel and H_1 a Struve function, while α is real. This integral equation arises in the problem of diffraction of a plane wave by a circular aperture in the case of normal incidence. The parameter α equals the product of wave number and radius of aperture. The integral equation of Magnus degenerates into an integral equation of the first kind as $\alpha\to\infty$. The solution of the latter can be given explicitly, and by a perturbation method the author obtains formulas for the solution of the original equation if α is large.

C. J. Bouwkamp.

Ahiezer, N. I.; and Ahiezer, A. N. On the diffraction of electromagnetic waves by a circular hole in a plane screen. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 53–56. (Russian)

In a previous paper, N. I. Ahiezer [same Dokl. (N.S.) **98** (1954), 333–336; MR **16**, 487; **17**, 498] obtained the solution of the integral equation

$$\int_0^\infty C(\lambda)J_m(\lambda r)\lambda^{m+1}d\lambda=0\ (n>a),$$

$$\int_0^\infty C(\lambda)J_m(\lambda r)\lambda^{m+1}\frac{d\lambda}{\sqrt{(\lambda^2-k^2)}}=F(r)r^m\ (0<r<a),$$

where $m\geqq 0$ is an integer, $k\geqq 0$, the square root is positive for $\lambda^2>k^2$ and has a negative imaginary part for $0<\lambda^2<k^2$, and the function $F(r)$ is sufficiently smooth on $0\leqq r\leqq a$. In the present paper this result is employed to derive an

iterative approximate method for the solution of the problem in the title. By way of an application, an approximate formula of C. Bouwkamp [Philips Res. Rep. **5** (1950), 321–332, 401–422; MR **12**, 774] is obtained as a consequence of the authors' method. *J. B. Diaz.*

Toraldo di Francia, G. Introduction to the modern theory of electromagnetic diffraction. Pubbl. Ist. Naz. Ottica, Firenze. Ser. II. no. **742** (1956), 55 pp.

This monograph is based upon some lectures in the theory of electromagnetic diffraction given at the International Summer Mathematics Conference in Varenna in 1956. According to the writer, the lectures complimented those given by C. J. Bouwkamp at the same conference. Only steady state phenomena are discussed. The material is presented in eight sections with the following outline: I. Representation of a scalar field in the presence of a finite obstacle or a screen with an aperture; II. The Kirchhoff theory with some critical remarks; III. Vector representation theorems including the Larmor-Tedone formulas; IV. The half-plane problem of Sommerfeld using the method of J. Brillouin (1949) for solution, with some comments regarding the implication of this solution in other diffraction problems of the same general class; V. Integro-differential equations of electromagnetic diffraction and the Babinet principle; VI. The notion of transmission and scattering cross section; VII. The variational method of Levine and Schwinger, including some comments on their formulation of the problem of diffraction of a plane electromagnetic wave by an aperture in an infinite conducting screen with an aperture; VIII. Diffraction of a screen with unidirectional conductivity. *A. E. Heins* (Pittsburgh, Pa.).

Phariseau, P. On the diffraction of light by progressive supersonic waves. Oblique incidence: intensities in the neighbourhood of the Bragg angle. Proc. Indian Acad. Sci. Sect. A. **44** (1956), 165–170.

In this paper the author shows that for high-frequency supersonic waves and angles of incidence in the neighborhood of the Bragg angle, the intensities of the orders 0 and -1 and of the orders $+1$ and -2 can be obtained from Raman and Nath's generalized theory [same Proc. **3** (1936), 119–125, 459–465]. The resulting expressions are the same as those obtained earlier by Bhatia and Noble using a different method [Proc. Roy. Soc. London. Ser. A. **220** (1953), 356–368, 369–385]. *C. H. Papas.*

du Castel, François. Journées d'études sur le développement des applications des interférences. XXI. L'introduction d'une longueur d'onde d'espace dans des problèmes de propagation radioélectrique. Rev. Opt. **35** (1956), 657–666 (1957).

Bates, R. H. T.; and Elliott, J. The determination of the true side-lobe level of long broadside arrays from radiation-pattern measurements made in the Fresnel region. Proc. Inst. Elec. Engrs. C. **103** (1956), 307–312.

Agostinelli, Cataldo. Su alcuni moti magneto idrodinamici in una massa fluida cilindrica rotante interessanti la Cosmogonia. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 479–508.

The author considers the equations of magneto-hydrodynamics in an incompressible viscous electrically conducting fluid, viz.

$$\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}-\Delta_2\mathbf{H}+\frac{4\pi\sigma\mu}{c^2}\left\{\frac{\partial\mathbf{H}}{\partial t}+\operatorname{curl}(\mathbf{H}\times\mathbf{v})\right\}=0,$$

$$\frac{\partial\mathbf{v}}{\partial t}-\mathbf{v}\times\operatorname{curl}\mathbf{v}+\frac{\mu}{4\pi\rho}\mathbf{H}\times\operatorname{curl}\mathbf{H}+$$

$$\operatorname{grad}\left(\tfrac{1}{2}v^2+\frac{p}{\rho}-U\right)-\nu\Delta_2\mathbf{v}=0,$$

$$\operatorname{div}\mathbf{v}=0,\quad \operatorname{div}\mathbf{H}=0,$$

in the usual notation. He uses cylindrical coordinates (r, φ, z), and seeks solutions independent of z. He is concerned, not with small oscillations about a steady state, but with rigorous solutions of this non-linear system.

It is readily seen that there exist functions V and W such that

$$H_r=\frac{1}{r}\frac{\partial V}{\partial\varphi},\ H_\varphi=-\frac{\partial V}{\partial r},\ v_r=\frac{1}{r}\frac{\partial W}{\partial\varphi},\ v_\varphi=-\frac{\partial W}{\partial r}.$$

It is shown that there are there types of solution of the form

$$V=-\tfrac{1}{2}h_0r^2+\Psi(r,\varphi,t),\ W=-\tfrac{1}{2}\omega r^2+\alpha\Psi(r,\varphi,t),$$

where h_0, ω, α are constants; these arise when (i) $\Delta_2\Psi=0$, or (ii) $\alpha^2=\mu/(4\pi\rho)$ and

$$\Delta_2\left(\frac{\partial\Psi}{\partial t}-\nu\Delta_2\Psi+(\omega-\alpha h_0)\frac{\partial\Psi}{\partial\varphi}\right)=0,$$

or (iii) $\Delta_2\Psi=k\Psi$ (k constant) and

$$\frac{\partial\Psi}{\partial t}+\left(\omega-\frac{\mu h_0}{4\pi\rho\alpha}\right)\frac{\partial\Psi}{\partial\varphi}-\nu k\Psi=0.$$

The solutions obtained are Fourier series of period 2π in φ, but the details are too complicated to give here.

E. T. Copson (St. Andrews).

★ **Belevitch, Vitold. Théorie des circuits de télécommunication.** Librairie Universitaire, Louvain, 1957. viii+384 pp.

This is essentially a text on classical network theory with brief excursions into the domains of nonlinear circuits, transient phenomena, amplifier theory and stochastic processes. The author devotes relatively little space to the establishment of a rigorous foundation theory and concentrates on the exposition of those techniques which, in his opinion, are of "real" value in practice. Much of the material is concerned with the analysis and design of tandem structures, with filter theory, in particular, being treated in considerable detail. The scope of the book may be judged by a partial listing of its chapter headings: Linear Systems, Energy Relations in Passive Circuits, Scattering Matrices, Image Parameters, Analytical Theory of Passive Circuits, Low-Pass Filters, Band-Pass Filters, Inductors and Transformers, Amplifiers, Transient Phenomena, Synthesis of Passive Networks, Bibliographical Notes.

Although the author's treatment of some basic topics is quite sketchy, his exposition is highly informative and marked by considerable originality. It is unlikely that this book will attain wide usage as a textbook for courses in network theory, but it is of great value as an up-to-date reference work for the circuit designer.

L. A. Zadeh (New York, N.Y.).

Povarov, G. N. The mathematical theory of synthesis of contact (1, k)-poles. Dokl. Akad. Nauk SSSR (N.S.) **100**, 909–912 (1955). (Russian)

A (p, q)-pole contact network is a net for which the pq admittances between the designated p terminals and the designated q terminals are considered. Estimates for the number of contacts required and the complexity of (p, q)-poles are given which are a generalization of the results of Shannon [Bell System Tech. J. **28** (1949), 59–98; MR **10**, 671]. A method for synthesizing (1, k)-pole networks called the cascade method is given and applied to a binary comparison circuit. *C. Saltzer.*

★ **Povarov, G. N. Mathematical theory of the synthesis of (1, k) terminal contacts.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 7 pp. $3.50.

Translation of the paper reviewed above.

Nöske, Heinz. Zum Stabilitätsproblem beim Abschalten kleiner induktiver Ströme mit Hochspannungsschaltern. Arch. Elektrotech. **43** (1957), 114–133.

Ledinegg, E. Schaltungstheorien im Zentimeter-Wellengebiet. Österreich. Ing.-Arch. **11** (1957), 20–36.

Lupanov, O. B. On possibilities of synthesis of networks of diverse elements. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 561–563. (Russian)

Bounds for the number of contacts required for the synthesis of arbitrary Boolean functions of n variables were given by Shannon [Bell System Tech. J. **28** (1949), 59–98; MR **10**, 671]. If in addition to relay contacts, other elements such as electronic switching devices are used, can this lower bound be reduced? By the study of an abstract formulation of this problem it is shown that no significant improvement is possible. *C. Saltzer.*

See also: Garstens, p. 805; Cyrlin, p. 828; Penndorp, p. 830; Lavine, p. 831; Bloch, p. 836; Aržanyh, p. 837; Rikitake, p. 858.

Thermodynamics and Heat

Spalding, D. B. A theory of inflammability limits and flame-quenching. Proc. Roy. Soc. London. Ser. A. **240** (1957), 83–100.

The energy and diffusion equations are solved for a one-dimensional, laminar, steady combustion wave in a pre-mixed gas which is in heat-transfer communication with solid surroundings. It is shown that in general a given combustible mixture has two possible flame speeds, but the lower one represents a normally unstable condition. If the heat-loss parameter increases, the two flame speeds first become coincident and then imaginary. The condition of coincidence is identified with the inflammability limit. Particular examples are evaluated in which heat loss is in turn by radiation and by conduction to the walls of a tube. *From the author's summary.*

★ **Popoff, Kyrille. Les bases mathématiques de la théorie des processus thermodynamiques irréversibles.** Mémor. Sci. Phys., no. 63. Gauthier-Villars, Paris, 1956. 85 pp. 1000 francs.

In this small monograph the author explains his own ideas on thermodynamics of irreversible processes. As usual, the entropy near equilibrium is written as a quadratic expression, $S=S_{eq}-\frac{1}{2}\sum g_{ik}x_ix_k$, where $x_1,\cdots,x_n$ are parameters describing the (macroscopic) state of the system. The 'affinities' or 'forces' are defined by $X_i=\sum g_{ik}x_k$. Now, Onsager writes for the time variation $\dot{x}_i=\sum L_{ik}X_k$, thus introducing a second matrix L_{ik}, and proved on statistical grounds $L_{ik}=L_{ki}$. The present author holds the view that the L_{ik} are not independent of the g_{ik}, but are uniquely determined by them.

The author postulates for the time variation of the system the equations $\ddot{x}_i=X_i$; that is, he considers (without bringing forward a justification) the X_k as 'forces' of Newtonian type. In addition it is postulated that of the $2n$ independent solutions only those have physical meaning that tend to zero for $t\to+\infty$. These n solutions can easily be shown to satisfy a set of first-order equations $\dot{x}_i=\sum L_{ik}^*X_k$, the coefficients L_{ik}^* being connected with the matrix g_{ik} in the following way. Let U be the orthogonal matrix of eigenvectors of g and Λ the diagonal matrix of its eigenvalue, so that $U^{-1}gU=\Lambda$. Then one easily finds $L=-U\Lambda^{-\frac{1}{2}}U^{-1}$. As this matrix is obviously symmetrical the author has derived Onsager's relations from the above postulate. (Incidentally, it may be noted that the relaxation times turn out to be the eigenvalues of g to the power $-\frac{1}{2}$; hence they depend on the special choice of the variables x_i.)

As an example the exchange of heat between two bodies with different initial temperatures is considered. The theory leads to the result that the rate of heat transfer depends solely on the specific heat and the geometry of the bodies. Other examples considered are osmosis and dissociation. *N. G. van Kampen* (Utrecht).

Trostel, R. Instationäre Wärmespannungen in einer Hohlkugel. Ing.-Arch. **24** (1956), 373–391.

A hollow sphere of homogeneous, isotropic, elastic material is, initially, completely insulated against thermal conduction and radiation from its surroundings, being at a uniform temperature and free from stress throughout. Suddenly this thermal insulation is removed, and unsteady thermal and stress fields occur in the sphere, these fields being due to heat exchanges between the material of the sphere and the media in contact with it over the inner and outer surfaces. The present paper gives an extremely detailed and full account of general and particular aspects of this problem, subject to conditions of quasi-static equilibrium and axial symmetry. The usual simplifying assumptions of linear elastic behaviour and linear surface thermal radiation are adopted. The boundary conditions are unsteady, and in general the (uniform) temperatures of the two media in contact with the sphere vary, not only in planes through the axis of symmetry, but also with time. The mathematical analysis required is reasonably straightforward but is somewhat lengthy; the appropriate background is provided by E. Melan and H. Parkus, Wärmespannungen infolge stationärer Temperaturfelder [Springer, Wien, 1953; MR **16**, 306]. The procedure is summarized as follows. First, the method of separation of variables is employed to provide series solutions of the heat conduction equation. Next, a particular integral of the thermal-elastic stress equations is found by means of a method involving the use of the so-called thermal-elastic displacement potential. The thermal-elastic stress field obtained in this way normally violates the required boundary conditions of zero applied stress. This particular stress field is therefore

corrected by subtraction of the elastic stress field directly set up by the boundary stresses associated with the former field, use being made of Love's displacement function. The paper discusses the problem of spherical symmetry as a special case, and some approximate formulae are derived for stresses at the inner and outer surfaces. Finally, reverting to conditions of axial symmetry, the much simpler special case of a thin shell is briefly discussed. The paper presents no numerical results.

H. G. Hopkins (Sevenoaks).

Friedmann, Norman E. The truncation error in a semi-discrete analog of the heat equation. J. Math. Phys. 35 (1956), 299–308.

The author deals with the simulation of the problem $\alpha u_{xx}=u_t$, $u(0,t)=F$, $u(L,t)=0$, $u(x,0)=0$ for $0<x\leq L$, $t>0$, with α and F constant, by the problem $\alpha\delta_x^2 v=h^2 v_t$, $v(0,t)=F$, $v(L,t)=0$, $v(x_k,0)=0$, where $h=L/N$ and $x_k=kh$ ($k=1,\cdots,N$), and where $\delta_x^2 v$ is the second central x-difference of v with spacing h. By exploiting known analytical expressions for $u(x_k,t)$ and $v(x_k,t)$, he obtains an upper bound on the error $|u-v|/F$ as a function of $(\alpha t)^{\frac{1}{2}}/L$ and N, independently of k, and displays the result graphically for several values of N.

F. B. Hildebrand (Cambridge, Mass.).

See also: Douglas, p. 827; Tawakley, p. 835; Bloch, p. 836; Batchelor, p. 843; Seiden, p. 854.

Quantum Mechanics

Yaglom, A. M. Application of function space integrals to the evaluation of the statistical sum of quantum statistics. Teor. Veoyatnost. i Primenen. 1 (1956), 161–167. (Russian. English summary)

The author makes use of the fact that the fundamental solution of a parabolic partial differential equation, such as the Schrödinger equation, may be expressed as a Wiener integral. By expanding the functional in the Wiener integral he obtains an expansion of the partition sum of quantum statistical mechanics in powers of Planck's constant. The calculation of successive terms in the series then reduces to evaluating successive higher moments of the Wiener process for which all odd power moments are zero and the even moments are expressible in terms of the variance. The terms in the series agree with those gotten earlier by other methods [e.g. E. P. Wigner, Phys. Rev. (2) 40 (1932), 749–759].

D. Falkoff (Waltham, Mass.).

Vachaspati. Quantum mechanics in generalized Hilbert space. Mat.-Fys. Medd. Danske Vid. Selsk. 30 (1956), no. 21, 28 pp.

An attempt is made to generalize the Hilbert space of quantum mechanics in analogy with the development of the general relativity theory from the theory of special relativity. The state vectors, ψ, $\bar{\psi}$, of quantum mechanics are found to be analogous to the four-velocity, v^μ, of relativity and therefore coordinates, χ, $\bar{\chi}$, are introduced corresponding to the coordinates x^μ of a particle, such that the time derivatives of χ and $\bar{\chi}$ equal ψ and $\bar{\psi}$. The metric η, used in constructing the probability density, is supposed to be a function of χ and $\bar{\chi}$. The unitary transformations of the usual theory are replaced by quite general transformations χ and $\bar{\chi}$. A tensor calculus for this generalized Hilbert space is developed and equations of motion for the states and the dynamical variables are postulated as generalizations of the usual Heisenberg equations when the ordinary time differentiation is replaced by invariant time differentiation. In this way a non-linear theory is obtained. However, the expectation values of the dynamical variables are found to be the same in the new theory as in the old, showing that this theory cannot give any physical results different from those of the usual theory. (Author's summary.)

M. J. Moravcsik (Upton, N.Y.).

Zaikov, Raško. Symmetrische Form der Quantenmechanik des Elektrons. Izv. Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. 5 (1955), 3–26. (Bulgarian. Russian and German summaries)

A. Proca has proposed a form of the Dirac equation for a free electron [Ann. Physique (10) 20 (1933), 347–440] with the proper time as a variable conjugate to the rest mass. In the present paper , this form of equation is extended to the case of an electron in an electromagnetic and a meson field, and a five-dimensional formalism is built up.

N. Rosen (Haifa).

McCarthy, I. E. Analytical solution of the covariant meson nucleon integral equation. Nuovo Cimento (10) 4 (1956), 991–1008.

A Tamm-Dancoff 2-meson approximation to $T=3/2$ meson-nucleon scattering is investigated by the Fredholm method. It is assumed that the zeros of the Fredholm denominator account for the observed resonances. A power series expansion in the coupling constant to evaluate these zeros yields a bound state for $g^2/4\pi\approx 14$ at 195 Mev, as well as a second unobserved resonance. Neither the accuracy of the above approach, nor that involved in estimating radiative corrections from scattering data, is discussed. In particular, use is made of erroneous pair-suppression arguments.

S. Deser (Copenhagen).

Groenewold, H. J. Quasi-classical path integrals. Mat.-Fys. Medd. Danske Vid. Selsk. 30 (1956), no. 19, 36 pp.

Quasi-classical path integrals can be used in quantum mechanics either for exhibiting the analogy with classical mechanics or to create a useful approximation process. The present paper gives some examples of non-relativistic motions with "semi-classical" zero order terms. The expansions are expressed with the help of quasi-classical paths. The connection of this scheme with the BWK approximation is particularily emphasized, although in the case of singularities a careful analysis is needed. The paper is also relevant to the unsolved problem of how to deal with competing classical paths.

M. J. Moravcsik.

Dalgarno, A.; and Stewart, A. L. On the perturbation theory of small disturbances. Proc. Roy. Soc. London. Ser. A. 238 (1956), 269–275.

The authors re-examine the expressions which standard Rayleigh-Schrödinger perturbation theory yields for the wave function and expectation values of dynamical variables in a perturbed quantum mechanical system. By elementary rearrangements the $(2s+1)$st approximation, $s=0, 1, 2, \cdots$, to the energy is expressed in a form which involves only the wave functions in sth approximation. Similar expressions are indicated for the expectation values of other operators. Finally, it is suggested that a practical approximate method for computing perturbed expectation values is obtained by applying variational methods to get approximate values for the

expressions given by perturbation theory.
A. S. Wightman (Princeton, N.J.).

de Broglie, Louis. Illustration par un exemple de la forme des fonctions d'ondes singulières de la théorie de la double solution. C. R. Acad. Sci. Paris **243** (1956), 617–620.

In his theory of the double solution the author has proposed that the wave function representing a particle must satisfy a partial differential equation which, though non-linear, is approximately linear in the sense that its only non-linear term is of negligible magnitude at distances r from the center of the particle which are appreciably greater than the particle radius a. The wave function may be written as the sum of two terms, one of which is like the solution of a linear wave equation, the other term being of negligible magnitude for values of r much greater than a. In the present note he shows by an explicit mathematical example that this situation may actually arise. It is not claimed, however, that either the equation or its solution corresponds to an actual physical particle. The example does show that the author's proposed theory is mathematically realizable. *O. Frink.*

Brodskiĭ, A. On the general theory of scattering of mesons. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 787–790. (Russian)

The theory of scattering of mesons by nucleons is formulated in terms of renormalized Green's functions, following the representation of the latter functions given by H. Lehmann [Nuovo Cimento (9) **11** (1954), 342–357; MR **17**, 332] and by M. Gell-Mann and F. E. Low [Phys. Rev. (2) **95** (1954), 1300–1312; MR **16**, 315].
E. L. Hill (Minneapolis, Minn.).

Jakobi, Georges. Etude du fluide de Proca. C. R. Acad. Sci. Paris **244** (1957), 1014–1016.

The equations describing a Proca fluid are taken to be

$$\dot{\Psi}=Dp_\alpha\gamma^\alpha\Psi$$

and the equations adjoint to these, where Ψ is a four-component spinor, the dot represents differentiation with respect to proper time, the γ^α are the Dirac matrices, p_α are the components of a time-like vector and D^2 is the length of the current vector formed from the spinor Ψ. This note contains results on the behavior of the derivatives of the various tensors, vectors and scales that can be constructed from the spinor Ψ. *A. H. Taub.*

Shibata, Takashi; and Kimura, Toshiei. Spin-orbit interaction energy of an electron based on the new fundamental group of transformations. J. Sci. Hiroshima Univ. Ser. A. **20** (1956), 37–45.

The first-named author has previously introduced a three parameter group of Lorentz transformations [same J. **18** (1955), 391–398; MR **17**, 202] in which there is involved a constant null-vector. In this paper the product of two such transformations is studied and the results applied to the Thomas effect in spin-orbit interaction. It is found that to obtain the correct results one must assume that the "constant" null vector must have its spatial components parallel to the variable spin-vector fo the electron moving about the nucleus. *A. H. Taub.*

Kanazawa, Hideo. On the spin-orbit interaction and the theory of plasma oscillations. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **6** (1956), 23–28.

It is shown that, in the case of an assembly of electrons, the spin-orbit interaction energy can be expressed as a sum of two terms, corresponding to the long-range and short-range parts of the Coulomb interaction, according to the collective description of D. Bohm and D. Pines [Phys. Rev. (2) **92** (1953), 609–625]. *N. Rosen* (Haifa).

Kalitzin, Nikola St. Über einige neue Methoden zur Beseitigung der Divergenzen in der Quantenelektrodynamik. Izv. Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **5** (1955), 37–66. (Bulgarian. Russian and German summaries)

Through the introduction of two sets of field variables to describe each kind of field, a theory is obtained which is invariant with respect to a change in sign of energy and mass. On the basis of several postulates it then follows that the energy, momentum and charge of the vacuum vanish, while all the well-known results of the usual quantum electrodynamics are obtained. *N. Rosen.*

Höhler, Gerhard. Wechselwirkung eines nichtrelativistischen Teilchens mit einem skalaren Feld für mittelstarke Kopplung. II. Z. Physik **146** (1956), 372–388.

[For part I see Z. Physik **140** (1955), 192–214; MR **17**, 114.] For an electron in a polar crystal, Pekar [Investigations on the electron theory of crystals, Gostehizdat, Moscow, 1951] used the product of a wave-function for the electron in a field due to the lattice in a polarised state, and a wave-function for the lattice polarised by the charge distribution of the electron. Here a variation principle is used with a Hamiltonian in terms of creation and annihilation operators for the optical vibrations. Attention is not restricted to the eigenvalue zero for the total wavenumber of electron + lattice. A scale variation parameter is introduced into the electron wave-function and the lowest energy state determined for a single product function. Next a linear combination of such product functions is used. The energy is expanded in descending powers of the coupling constant between electron and lattice (the strong coupling approximation). With mathematical detail the results are compared with those of Pekar and others. The mass of the "polaron" (the electron and its associated lattice deformation) is evaluated and discussed with reference to the work of others. Weak coupling is briefly considered. The effect of two variation parameters is discussed, viz. the scale-factor mentioned above and a factor allowing for the centre of the potential well for the electron being correlated with the state of the lattice. It is concluded that Pekar's polaron conception is useful only for strong coupling and that of Lee, Low and Pines [Phys. Rev. (2) **90** (1953), 297–302] and Gurari [Phil. Mag. (7) **44** (1953), 329–336] only for weak coupling.
C. Strachan (Aberdeen).

Valatin, J. G. On the functional approach to the quantum field equations. Nuovo Cimento (10) **4** (1956), supplemento, 726–732.

This is a brief account, presented at a conference, of some work which was subsequently published in more intelligable detail [Proc. Roy. Soc. London. Ser. A. **229** (1955), 221–234; MR **17**, 332]. A classical field theory with external sources is quantized by replacing the field variables in the Lagrange equations by functional derivatives with respect to the sources. These operate on a functional of the sources which is a highly generalized "wave function". From this general approach the author is able to correlate the work of a variety of authors who developed different aspects of the algebra of variously

defined expectation values of products of Heisenberg operators. *P. T. Matthews* (Rochester, N.Y.).

Watson, K. M. Multiple scattering by quantum-mechanical systems. Phys. Rev. (2) **105** (1957), 1388–1398.

A simple description is given of the quantum-mechanical theory of multiple scattering. The separation of the scattered wave into "coherent" and "incoherent" parts is discussed in greater generality than has been done previously. Applications to transport theory are described. Specific calculations are made of the refractive index of a medium which is "polarized" by the scattered particle (Lorentz-Lorenz formula) and also of a medium which has correlated structure (critical opalescence). Other applications are given. *Author's summary.*

Nakanishi, Noboru. General integral formula of perturbation term in the quantized field theory. Progr. Theoret. Phys. **17** (1957), 401–418.

The author reconsiders the theorem of Dyson in quantum field theory [Phys. Rev. (2) **75** (1949), 1736–1755; MR **11**, 145] that it is possible to test the ultra-violet convergence of a Feynman graph by counting the powers of momentum in the denominator that arise from the propagators. The general nth order diagram is considered. The denominators are first combined by making use of the parametric representation of Feynman [ibid. **76** (1949), 769–789; MR **11**, 765] and the momentum integrals performed. The resultant parametric integrations are then examined. By analysing the singularities that appear in the remaining integrations over the Feynman parameters, the theorem is established. The derivation given has the advantages of being more elegant than Dyson's original proof as well as being formally covariant at each stage of the calculation. The author also claims that Dyson's procedures do not necessarily lead to a unique result for the integrations.

The general formula for the nth order graph obtained is then applied to the Δ_F' and S_F' functions. By an appropriate change of variables, a perturbation expression of Lehmann's spectral functions is obtained [Nuovo Cimento (9) **11** (1954), 342–357; MR **17**, 332.] The spectral form for the "three point" Green's function derived in this fashion is more general than that postulated by Nambu [Phys. Rev. **100** (1955), 394–411; MR **17**, 440.]

The general formula is also applied to the question of the convergence of the perturbation expansion for the theory with a $\lambda\phi^3$ interaction. *R. Arnowitt.*

Kanki, Takeshi; Murata, Koichi; and Sunakawa, Sigenobu. The functional integrals in quantum theory. Progr. Theoret. Phys. **17** (1957), 7–18.

The direct application of Feynman's "sum over histories" method to field theory leads to the notion of a functional integral over anti-commuting functions for the fermi particles. This has always been avoided by Feynman himself. It was shown by the reviewer and Salam [Nuovo Cimento (10) **2** (1955), 120–134; MR **17**, 643] how this integral could be carried out, for the field propagators which involve vacuum expectation values between the vacuum states at $\pm\infty$. This paper shows that the method is not applicable to more general expectation values, and also that an adiabatic switching off the interaction in the initial and final states was implicitly assumed. *P. T. Matthews* (Rochester, N.Y.).

Clementel, E.; and Villi, C. On a new nucleon-nucleon potential. Nuovo Cimento (10) **4** (1956), 935–939.

The authors investigate a lagrangian density which a) preserves the linearity of the field equation, b) suppresses high-momentum components of the field, and c) goes over into the usual lagrangian when the momentum cut-off tends to infinity. Their field equation is similar to one previously obtained by Bopp [Ann. Physik (5) **38** (1940), 345–384; MR **2**, 336]. The solution which asymptotically behaves like the usual one contains two parameters which are fixed by the cut-off momentum, one is essentially the pion Compton wave length, the other can be made equal numerically to the K-meson Compton wave length. Such a smaller distance, of the order of 0.2 to 0.5 of the pion Compton wave length, is known to be required in the present pion theory of nuclear forces and inside of which the structure of the nucleon field is unknown. The two-nucleon potential energy obtained by the authors contains a central part which for even singlet and triplet states becomes strongly repulsive at short distances. It also contains a tensor part which has the sign required by the deuteron quadrupole moment and which behaves as r^{-1} at short distances; it has the same mathematical structure as the tensor force in the Møller-Rosenfeld-Schwinger mixed meson theory.

J. Leite Lopes (Pasadena, Calif.).

Ikeda, Mineo; and Miyachi, Yoshihiko. On an extended framework for the description of elementary particles. Progr. Theoret. Phys. **16** (1956), 537–547.

The authors present a development of Pais's theory of ω-space [Physica **19** (1953), 869–887; MR **15**, 766], in which they find it necessary to introduce the b-field of Yang and Mills [Phys. Rev. (2) **96** (1954), 191–195; MR **16**, 432.] They also discuss the similarities between the b-field, the electromagnetic field and the gravitational field by following the treatment of the reviewer [ibid. **96** (1954), 1683–1685; MR **16**, 532]. *S. N. Gupta.*

Yennie, D. R.; and Suura, H. Higher order radiative corrections to electron scattering. Phys. Rev. (2) **105** (1957), 1378–1382.

The higher order radiative corrections are examined in the infrared region, and Schwinger's conjecture regarding the functional dependence of these corrections on the energy resolution is proved. *Author's summary.*

Gourary, Barry S.; and Adrian, Frank J. Approximate wave functions for the F center, and their application to the electron spin resonance problem. Phys. Rev. (2) **105** (1957), 1180–1192.

The vacancy model of the F center is treated by a simplified Hartree method. The ions are treated as point charges, and the potential of the lattice is computed. The simplified Hartree equation is solved variationally, and the electronic polarization is computed by a self-consistent method which takes account of the screening action of the F-center electron. The lattice distortion is then calculated. The resulting energies are compared with the available optical data. The agreement is good.

The hyperfine structure of the F center is computed, using a determinantal wave function. The predicted hyperfine splittings agree fairly well with the experimental results of Lord and Jen on the resolved hyperfine structure of LiF.

The effects of exchange and overlap are also discussed.

Authors' summary.

Gourdin, M. Diffusion nucléon-nucléon par des forces non centrales. II. Diffusion proton-proton. J. Phys. Radium (8) **18** (1957), 85–91.

The derivation of an explicit expression for the non-relativistic neutron-proton scattering cross section including tensor forces, which was reported earlier by the same author [same J. (8) **17** (1956), 988–996; MR **18**, 702], is here extended to proton-proton scattering. The Coulomb potential (for point nucleons) is included. The angular distribution is expressed as a pure Coulomb term and an expansion in Legendre polynomials. For the latter the coefficients as functions of the Coulomb and nuclear phase shifts and the admixture parameters are given for all terms with angular momenta $L \leqq 3$. *F. Rohrlich.*

Redmond, P. J. Heisenberg operators in a Lagrangian formalism. Phys. Rev. (2) **105** (1957), 1652–1655.

The problem of the formulation of field theories in terms of time-ordered products of Heisenberg operators has received some attention recently. Often, in these approaches, certain plausible assumptions have been made. The author derives some of the results obtained this way: the expressions for the S matrix and the reduction formulas found previously by Lehman, Symanzik and Zimmerman are derived from a Lagrangian formalism.

Consider a system consisting of fermions with field operators interacting with bosons with field operators; it is described by a Lagrangian which can be separated into three parts corresponding to free fields, an interaction term and a term containing external sources. Using the notation of Umezawa and Visconti [Nuovo Cimento (10) **1** (1955), 1079–1103; MR **17**, 443] the equations for the U-functions are derived; they are solved by the use of the perturbation theory expression for the S matrix.

It is shown that the problem reduces to finding an expression for the S matrix for free fields. First, the author considers free fields with sources. Starting from the formula for the action of a boson field with external sources and the equation of motion, the writer finds a solution of this system using the generating function for Wick's S product for the free fields and the causal Green's function with the Fourier expansion. The same procedure is applied to a system of non-interacting bosons and fermions with action.

The expression for the S matrix is in the form of an infinite series. As a side product a definition of the Feynman amplitudes is obtained.

Next, the author solves the problem of the interacting fields, deriving the equations for the U-functions and obtaining the S matrix by setting the source terms equal to zero. Thus the result as given by Lehman is obtained.

Finally, he derives the formulas for the matrix elements of operators in terms of vacuum T products and bare-particle matrix elements, called reduction formulas, obtained previously by Lehman, Symanzik and Zimmerman. *M. Z. Krzywoblocki* (Urbana, Ill.).

Jackson, J. D. On the use of the complete interaction Hamiltonian in atomic rearrangement collisions. Proc. Phys. Soc. Sect. A. **70** (1957), 26–33.

The paper deals with a problem which has been in the foreground in the past few years: the transfer of an electron from an ion to another in a collision process. It has been shown [e.g. D. R. Bates and A. Dalgarno, Proc. Phys. Soc. Sect. A. **65** (1952), 919–925] that the inclusion of the ion-ion interaction is important for getting good results. The present paper gives a physical explanation of why this is so. Furthermore the paper shows why the first Born approximation gives a good agreement with experiments on the capture of electrons by protons in hydrogen, but not with capture of electrons by alpha particles in hydrogen. Qualitatively, the latter effect can be explained by the fact that for heavy ions the rearranged system feels a long range Coulomb interaction whereas for hydrogen ions the final interaction is short range, similar to the initial dipole type interaction. The paper shows in detail that the second Born approximation practically vanishes for heavy ions. *M. J. Moravcsik* (Upton, N.Y.).

Seiden, Joseph. Réversibilité et irréversibilité en résonance nucléaire. I. Théorie de la relaxation nucléaire dans les liquides. J. Phys. Radium (8) **18** (1957), 173–192.

The author claims to obtain kinetic equations governing the approach to the equilibrium state for a system of nuclear spins in a liquid or lattice using the quantum mechanical equations of motion. He does indeed use the quantum mechanical Hamiltonian for the free spins. However, he replaces the unknown Hamiltonian for the lattice and spin-lattice interaction by a time-dependent random function $V(t)$. By making sufficient assumptions about the structure and averages of $V(t)$ and about the initial distribution of the spins, he is able to obtain Boltzmann-like equations for the relaxation process. *D. Falkoff* (Waltham, Mass.).

Placzek, G. Incoherent neutron scattering by polycrystals. Phys. Rev. (2) **105** (1957), 1240–1241.

General expressions are given for the first terms in the expansion in powers of the neutron to nuclear mass ratio of the total cross-section for incoherent scattering of neutrons by polycrystals. Special limiting cases of these expressions had been published earlier. *Author's summary.*

Valatin, J. G. Nucleon motion in a rotating potential. Proc. Roy. Soc. London. Ser. A. **238** (1956), 132–141.

The types of motion of a particle in a rotating oscillator potential are investigated. The hamiltonian in the rotating reference system in transformed into the sum of those of three independent harmonic oscillators; the angular momentum in the same system is calculated. These quantities are then obtained for an atomic nucleus which is approximated by a system of nucleons moving in such a slowly rotating potential. Expansion in a power series of the angular velocity introduces the moment of inertia. The angular momentum has positive and negative contributions from the particle orbits which are large compared with those from the rotation of the orbits. These orbital contributions cancel out near an equilibrium deformation which gives the deformed closed-shell core practically no angular momentum to couple with the outer particles. The relevant coupling is between these and the angular momentum of the collective rotation of the whole nucleus. *J. Leite Lopes.*

Marumori, Toshio; Suekane, Shôta; and Yamamoto, Atsuko. Nuclear deformability and shell structure. Progr. Theoret. Phys. **16** (1956), 320–340.

The nuclear Schrödinger equation is transformed into the collective representation. If the particle excitation frequencies are larger than those of the collective motion,

the variables can be separated and one obtains the surface harmonic oscillations. When the particle excitation frequencies are comparable to the collective frequencies, the nucleus is described in terms of a coupled system of collective and particle degrees of freedom. The Bohr-Mottelson hamiltonian is obtained. For a small deformation of the closed shell average potential the surface rigidity is calculated to second order by perturbation theory. The values obtained for closed shells are generally larger than those estimated from the hydrodynamical model. They are used to calculate the quadrupole moments of the "core $\pm$ one extra particle" nuclei, when the success and limitations of the theory are exhibited.
J. Leite Lopes (Pasadena, Calif.).

Mouhasseb, Adnan. Sur les vibrations collectives d'une structure "en couches" de particules. C. R. Acad. Sci. Paris **243** (1956), 1289–1292.

It is pointed out that for incomplete shells the coefficients B, associated with the mass transported by the collective flow in the Bohr-Mottelson model, can have values which are considerably larger than those corresponding to closed shells. *J. Leite Lopes.*

Kockel, B. Versuch einer halbklassischen nichtlinearen Theorie der Kernkräfte. Wiss. Z. Karl-Marx-Univ. Leipzig. Math.-Nat. Reihe **3** (1953/54), 401–404.

The solution of the non-linear equation:

$$\Delta V - k^2 V + l^2 V^3 = -4\pi\rho$$

which is finite everywhere and vanishes exponentially at infinity, contains terms which correspond to a many-body force. *J. Leite Lopes* (Pasadena, Calif.).

Alonso, Marcelo. Nuclear forces. Rev. Soc. Cubana Ci. Fís. Mat. **3** (1956), 179–230. (Spanish)

An exposition of the main properties of the two-nucleon system at low energies. *J. Leite Lopes.*

Kalitzin, Nikola St. Über die Wechselwirkung des Nukleons mit dem Mesonfeld. Izv. Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **5** (1955), 213–229. (Bulgarian. Russian and German summaries)

The motion of a nucleon is described by means of a wave equation involving six coordinates and eight-rowed matrices. The meson field is described by means of a third-rank antisymmetric tensor potential satisfying linear field equations. It is expected that this quantum meson-dynamics will be renormalizable. *N. Rosen.*

Basile, Robert. Sur une nouvelle méthode de calcul des sections efficaces des réactions nucléaires provoquées par des rayonnements γ de freinage. C. R. Acad. Sci. Paris **243** (1956), 1759–1761.

An empirical formula is proposed for the bremsstrahlung spectrum, which enables one to give an explicit expression for the induced photo-nuclear cross-sections. The method avoids the use of recurrence formulas used in the photo-difference method. *P. T. Matthews.*

Church, E. L.; and Weneser, J. Effect of the finite nuclear size on internal conversion. Phys. Rev. (2) **104** (1956), 1382–1386.

Nuclear size may affect the emission of electrons through internal conversion either by changing the radial dependence of the electron wave functions, or by the penetration of the nuclear charge and current distribution by the atomic electrons. The transverse photon may then pass from the electron to the nucleus, rather than vice versa. The latter effect depends on nuclear matrix elements, which are different from those for γ-ray emission of the same multipole order. These supply new information about nuclear structure and are of particular interest when the corresponding gamma matrix element vanishes.
P. T. Matthews (Rochester, N.Y.).

Tamura, T. On the collective description of nuclear surface oscillation. Nuovo Cimento (10) **4** (1956), 713–735.

The general method of redundant coordinates, subsidiary conditions and unitary transformations, introduced by Miyazima and Tamura [Progr. Theoret. Phys. **15** (1956), 255–272] is applied to the individual particle Hamiltonian to express it in terms of collective rotations, vibrations and (strong) coupling terms. The results on rotational motion agree with those of Lipkin, de Shalit, and Talmi [Nuovo Cimento (10) **2** (1955), 773–798; MR **17**, 692] and the two dimensional Hamiltonian for rotations and vibrations is the same as that of F. Villars (unpublished), but the author claims his subsidiary condition is easier to handle. *P. T. Matthews.*

Mittelstaedt, P. Zur theoretischen Bestimmung der Neutronenreaktionsquerschnitte nach dem optischen Kernmodell. Z. Naturf. **11a** (1956), 663–676.

The optical model based on energy independent square well potentials, which are in agreement with experiment for energies below 3 Mev, do not fit the data in the 3–14 Mev region. The effects of a diffuse nuclear surface, and energy dependent potentials are examined. It is found that the theory can be brought into reasonable agreement with experiment by including the energy dependence of the potentials to be expected from the effect of the Pauli principle on the mean free path of a neutron passing through nuclear matter. *P. T. Matthews.*

Arai, Tadashi. New approach to the quantum-mechanical analysis of the electronic structures of molecules. The method of deformed atoms in molecules. J. Chem. Phys. **26** (1957), 435–450.

The method of atoms in molecules introduced by Moffitt [Proc. Roy. Soc. London Ser. A. **210** (1951), 245–268] is generalized to take account of the deformation of atoms during molecular formation. This generalization increases the accuracy of the method and also simplifies the calculations, since the same orbital functions may now be used to approximate neutral and ionic states of a given (deformed) atom. For many-electron systems computational difficulties arise from higher order permutations. These difficulties may be avoided by an additional approximation involving an interaction operator. The resulting theory is equivalent to that studied by the reviewer [Proc. Phys. Soc. Sect. A. **69** (1956), 49–56, 767–776]. *A. C. Hurley* (Melbourne).

Bingel, W. Eine neue Methode zur Berechnung der Elektronenterme von Molekülen. Z. Naturf. **12a** (1957), 59–70.

Der RaC—RaC′-Zerfall wurde durch β—γ- und e^-—β-Koinzidenzmessungen untersucht. Bei den β—γ-Koinzidenzmessungen wurden in Koinzidenz mit γ-Strahlung der Energie $>$ 0,8 MeV β-Gruppen mit den Grenzenergien (1,45$\pm$0,05) MeV, (1,1$\pm$0,1) MeV und etwa 0,5 MeV ge

funden. Mit energieärmerer γ-Strahlung koinzidiert zusätzlich eine β-Gruppe von $(1{,}9\pm0{,}1)$ MeV. Wenn eine β-Gruppe von etwa 2,6 MeV vorhanden ist, hat sie eine Intensität $<4\%$. Bei den e^-—β-Koinzidenzmessungen wurde in Koinzidenz mit den K-Konversionselektronen der vollständig konvertierten 1.416-MeV-γ Strahlung eine β-Gruppe von $(1{,}8\pm0{,}2)$ MeV beobachtet, in Koinzidenz mit den K-Konversionselektronen der 609-keV-γ-Strahlung eine β-Gruppe von $(1{,}5\pm0{,}2)$ MeV. Ein Zerfallsschema für den RaC—RaC'-Zerfall wurde aufgestellt.
Author's summary.

Hamaguchi, M. On the hydrodynamical model in multiple production of mesons. Nuovo Cimento (10) **4** (1956), 1242–1261.

A viscous fluid model is developed, in place of Landau's perfect fluid model [Izv. Akad. Nauk. SSSR Ser. Fiz. **17** (1953), 51–64], using relativistic hydrodynamics. The effect of viscosity is treated as a perturbation. The result is that the energy distribution is slightly decreased, the scattering angle and the number of particles somewhat increased, compared with Landau's model.
P. T. Matthews (Rochester, N.Y.).

Senitzky, I. R. Quantum effects in the interaction between electrons and high-frequency fields. Vacuum fluctuation phenomena. Phys. Rev. (2) **104** (1956), 1486–1491.

The direct calculation of the dispersion in the electron velocity caused by vacuum fluctuations leads to an infrared divergence. By applying the appropriately modified Bloch-Nordsieck transformation to eliminate the effects of low energy photons outside the cavity, a finite result is obtained. *P. T. Matthews* (Rochester, N.Y.).

Livšic, M. S. On the intermediate system formed in the scattering of elementary particles. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 799–802. (Russian)

The author proposes to describe the scattering of two elementary particles, a_1 and a_2, by a process of the form $a_1+a_2\to C\to a_2+a_1$, in which there is assumed to be a definite "intermediate" state C of definable characteristics. The argument is based on the analogy of the theory of photon scattering by electrons as an absorption of the photon follows by its re-emission. The calculation depends on a treatment of the scattering matrix given by the author in an earlier paper [M. S. Livshits, Ž. Eksper. Teoret. Fiz. **31** (1956), 121–131].

The theory leads to the result that for two spinless particles of masses m_1 and m_2, the intermediate state being an s-state, the energy spectrum of the state C is continuous, covering the intervals

$$-(m_1+m_2)\leq\lambda\leq-|m_1-m_2|,\ |m_1-m_2|\leq\lambda\leq m_1+m_2.$$

State C appears to be stable against emission of the same particles unless the mass of one of them is zero. A brief discussion is given for the case that one of the particles has spin $\frac{1}{2}$, and although detailed conclusions are not stated they would seem to be similar to that for the spinless case. It is concluded that in the state C the system can move like a particle with a continuous internal energy spectrum. *E. L. Hill.*

Klein, Abraham; and McCormick, B. H. Construction of the adiabatic nuclear potential: formalism. Phys. Rev. (2) **104** (1956), 1747–1757.

A new formalism is given for the construction of the two-nucleon potential. It involves an expansion in the number of virtual mesons exchanged and permits the exact treatment of the nucleon self field. The potential is obtained from the R-matrix by the Lippmann-Schwinger integral equation, which requires off-the-energy shell potential matrix elements in the second order. These are determined from the Schrödinger equation. The potential is meaningful only for distances larger than about $\frac{1}{2}(\hbar/\mu c)$ and involves the exchange of one and two mesons. It is computed for the P-wave meson coupling and depends on the renormalized coupling constant, the nucleon source function and the total cross sections for pion-nucleon scattering. *J. Leite Lopes.*

Loinger, Angelo. Sull'elettrodinamica classica dell'elettrone puntiforme. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 309–313.

The divergences of the electron self-field are eliminated if this is defined as half the difference between the retarded and the advanced field [Dirac, Proc. Roy. Soc. London. Ser. A. **167** (1938), 148–169; Ann. Inst. H. Poincaré **9** (1939), 13–49; MR **1**, 94]. In this paper, the advanced field is omitted as non-physical. Of the divergent terms in the retarded field at the electron world line, one is inertial and can be incorporated in the electron mass, as well known. The author considers as evident that the terms which are proportional to the radius of the tube around the world line should be omitted from the self-force. *J. Leite Lopes* (Pasadena, Calif.).

See also: Lehner, p. 808; Bloch, p. 836; Nagy, p. 856.

Relativity

Karpman, Gilbert; et Raman, Varadaraja Venkata. Sur une généralisation possible de la théorie des fluides à spin de Weyssenhoff. C. R. Acad. Sci. Paris **243** (1956), 1284–1287.

The equations of motion are obtained in the relativistic theory of fluids with spin, for the case in which the spin density has time components in the proper system.
J. Leite Lopes (Pasadena, Calif.).

Nagy, K. Über die Bewegungsgleichungen des Pol-Dipol-Teilchens. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 683–685.

The author outlines a method of deriving the equations of motion of a classical particle with spin which is based on Infeld's generalization of the derivation of the equations of motion of a particle in its own gravitational field [same Bull. **3** (1955), 213–216; MR **17**, 201]. Various consequences of these equations are deduced. *A. H. Taub.*

See also: Vachaspati, p. 851; Zaikov, p. 851; Jakobi, p. 852; Shibata and Kimura, p. 852.

Astronomy

Nobile, Vittorio. Il problema del riferimento pei moti stellari e la sua essenziale connessione con quello della ricerca del potenziale galattico. La soluzione rigorosa del complesso dei due. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 715–719.

The author is disappointed by the fact that astronomers

have ignored a paper he published in 1928 [Mem. Soc. Astr. Ital. (N.S.) **4** (1928), no. 2], in which he showed that the problem of stellar motions in the Galaxy is tied up with the problem of the earth's rotation and gave directions on how to solve the impasse. This neglect has prompted him to publish the present note, which is only a summary of his earlier paper, in the hope that this time it will be noticed, and to repeat his warnings to astronomers on how useless all their work on galactic dynamics is, unless they mend their ways and follow his directions.

The author displays a lack of appreciation of modern work in positional astronomy. Astronomers have been aware all along of the problem he believes he has discovered and the question of a suitable frame of reference has been given a great deal of attention at international meetings; moreover, they have found practical solutions to the problem and have put them into effect.

L. Jacchia (Cambridge, Mass.).

Merman, G. A.; und Kotschina, N. G. Über die Grenze des Einfanggebietes im restringierten hyperbolischen Dreikörperproblem. Byull. Inst. Teoret. Astr. **6** (1956), 349–377. (Russian. German summary)

Baženov, G. M. The first-order perturbations of the mean motion of an infinitesimal body in the problem of three bodies. Byull. Inst. Teoret. Astr. **6** (1956), 378–407. (Russian. English summary)

Grémillard, Jean. Sur la recherche de certaines solutions périodiques du problème des trois corps à inclinaison quelconque. C. R. Acad. Sci. Paris **244** (1957), 1011–1014.

Let n and n' denote the mean motions in the two osculating orbits at $t=0$ about the central body in the three-body problem. Further, let it be assumed that $n/n'=p/q$, where p and q are relatively prime positive integers and $q<p$. From certain results of von Zeipel [Nova Acta Soc. Sci. Upsal. Sect. I (3) **20** (1904), no. 9], the author deduces a method of searching for periodic solutions of the third kind, when the mutual inclination of the planes of the two osculating orbits is an angle between 0° and 180°, and the difference $p-q$ is an even number. Several cases are possible according as certain hessians are zero or are different from zero [cf. also Grémillard, C. R. Acad. Sci. Paris **234** (1952), 2339–2341; **239** (1954), 153–155; MR **13**, 996; **16**, 181].

E. Leimanis (Vancouver, B.C.).

Isvekov, V. A. On the accuracy of the determination of corrections of the elements in the case of improvement of minor planet orbits. Byull. Inst. Teoret. Astr. **6** (1956), 416–422. (Russian. English summary)

Isvekov, V. A. An approximate method of the control of observations in the case of the improvement of orbits of minor planets. Byull. Inst. Teoret. Astr. **6** (1956), 423–427 (1 plate). (Russian. English summary)

Fesenkov, V. G. Some properties of motion of a gravitating body in a resisting medium. Astr. Ž. **33** (1956), 614–621. (Russian. English summary)

The author investigates the motion of a particle in the gravitational field of a sphere, having an atmosphere of homogeneous density. The drag-force is assumed to be proportional to the velocity relative to the atmosphere. The numerical integration shows that the orbit tends to become circular under the influence of the drag. The results obtained represent the first approximation for a rather difficult problem of the motion of a satellite in the resistant medium.

P. Musen (Cincinnati, Ohio).

Herovanu, Mircea. Méthode graphique pour l'étude de la diffusion de la lumière dans l'atmosphère. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 673–678. (Romanian. Russian and French summaries)

The author presents a graphical method for the determination of the diffusion of light in an anisotropic medium. The testing of the law of Mie seems to be the main object. {The author does not seem to be familiar with the existing literature pertaining to these types of problems. The fundamental contributions of Ambartsumyan, Chandrasekhar, Kourganoff, etc., are not even mentioned.}

K. Bhagwandin (Blindern).

★**Schatzman, Evry. Origine et évolution des mondes.** Editions Albin Michel, Paris, 1957. 404 pp. (4 plates). 1500 francs.

A semi-popular account, without mathematical details, of a large number of recent investigations.

Velghe, A. G. Studies of dark nebulae based on truncated distributions of stars. Astr. J. **61** (1956), 241–253.

A method is developed for the solution of the extinction problem, including the determination of the distance, the extinction power and the radial extension of a dark nebula. The method is intended for analyses of counts referring to classes of stars with small dispersion in absolute magnitude. The introduction of boundary conditions enables the computer to make use of interpolation formulae, in a first approximation of truncated distribution data. The adjustments are made through a numerical method. It is shown that integrated distributions can be used by themselves for solving the problem, without the assumption of a model of a dark nebula.

Author's summary.

See also: Backus, p. 847.

Geophysics

Pritchard, D. W. The dynamic structure of a coastal plain estuary. J. Marine Res. **15** (1956), 33–42.

Pritchard, D. W.; and Kent, Richard E. A method for determining mean longitudinal velocities in a coastal plain estuary. J. Marine Res. **15** (1956), 81–91.

In the first paper, the time mean equations of a coastal plain estuary are derived, and these are shown to be integrable under appropriate boundary conditions, giving the eddy flux of momentum. Using observations of temperature and salinity, to determine the pressure field, and of mean velocity and tidal velocity, the method is applied to the James River estuary. The results show that the mean acceleration terms are negligible, and that the eddy flux in the two directions appear to be proportional a result to be compared with that of Fleagle and Badgley for the atmosphere [Occasional Rep. Dept. Meteorol. and Climatol. Univ. of Washington no. 2 (1952)].

With this one assumption, and neglecting the mean acceleration terms, the second paper show how the equations of motion may be used to determine the mean longitudinal velocity, the values of which agree favourably with the observations.

D. C. Gilles (Manchester).

Veronis, G.; and Stommel, Henry. The action of variable wind stresses on a stratified ocean. J. Marine Res. 15 (1956), 43–75.

For a horizontal infinite rotating ocean composed of two homogeneous layers, the authors derive the equations of motion under wind stresses τ, neglecting horizontal friction but assuming some variation of the Coriolis parameter. The equations are then simplified by introducing one dependent variable, whence it is shown that two classes of solutions (baroclinic and barostrophic waves) exist.

Treating first the free motion, six possible kinds of waves are obtained. The application of a moving wind system $\tau=W\sin(lx+\nu t)$ causes waves of the same period and wavelength as the applied stress to occur, and resonance occurs in some interesting cases. A simple transient wind, the approximation of a storm, $\tau=0$, $t<0$; $\tau=W\cos lx$, $t>0$, gives rise to all six possible waves, and their relative magnitudes and importance for practical case are ex- amined.

Finally there is some examination of the neglected effects. *D. C. Gilles* (Manchester).

Freeman, John C., Jr.; and Baer, Ledolph. Pseudo-characteristics. Trans. Amer. Geophys. Union 38 (1957), 65–67.

The pseudo-characteristics are defined as the lines along which the dependent variables in non-homogeneous equations have the same relationship as the similar variables in the corresponding homogeneous equations. A sample computation of water levels caused by wind stresses, using the pseudo-characteristics, is included. *Authors' summary.*

Blinova, E. N. A method of solution of the non-linear problem of atmospheric motions on a planetary scale. Dokl. Akad. Nauk SSSR (N.S.) 110 (1956), 975–977. (Russian)

In the present paper the author investigates the non-linear problem of longtime weather forecasting, especially in the troposphere. The quantities relative to the vertical motion of the atmosphere are considered small with respect to the horizontal ones. She reduces the problem to the solution of a system of five non-linear partial differential equations. These equations are too complicated to be reproduced here. Thereafter, she introduces some remarkable approximations, which make the equations much more tangible. Finally, the solution is obtained, with specified boundary-conditions, in terms of two triple-integrals (with finite ranges). The terms appearing under the sign of the integrals are infinite series, the members of which comprise exponential- and spherical functions. On account of space, it is not possible to present her solution here. By knowing the initial-conditions for a certain function, it is possible to find all the necessary points on a semi-sphere. *K. Bhagwandin* (Oslo).

Morton, B. R. Buoyant plumes in a moist atmosphere. J. Fluid Mech. 2 (1957), 127–144.

This paper describes a simple model which can be used to investigate the transport of water vapour by thermal plumes in the atmosphere. For an approximate treatment of these plumes, it is assumed that the vertical velocity, temperature and specific humidity are constant across the ascending column, and that the inflow velocity due to mixing at the edge of the plume is proportional to the vertical velocity within the plume. The behaviour of the rising air is then investigated by means of equations representing the conservation of mass, momentum, heat and water vapour, and numerical solutions are obtained for representative cases. *From the author's summary.*

Gutman, L. N. Theoretical model of a tornado. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1957, 79–93. (Russian)

On the basis of a solution of non-linear equations of thermohydrodynamics of the atmosphere an idealized model of a tornado is constructed as of a process conditioned by the presence in the atmosphere of large rotational moments and also a sharply pronounced moisture stability. *Author's summary.*

Rikitake, Tsuneji. Stability of the earth's dynamo. Bull. Earthquake Res. Inst. Tokyo 34 (1956), 283–289. (Japanese summary)

★**Гордеев, А. В.; и Шарупич, С. Г. [Gordeev, A. V.; i Šarupič, S. G.] Уравнивание типовых фигур триангуляции [Graduation of typical figures of triangulation.] 2nd ed., revised and augmented.** Izdat. Geodez. Lit., Moscow, 1956. 195 pp. 8.30 rubles.

Hallert, B. Untersuchungen über die Genauigkeit des ersten Modelles einer Aerotriangulation. Schweiz. Z. Vermessg. Kulturtech. Photogr. 55 (1957), 47–52, 74–79.

★**Łomnicki, Antoni. Kartografia matematyczna. [Mathematical cartography.] 2nd ed.** Państwowe Wydawnictwo Naukowe, Warszawa, 1956. 176 pp. zł. 16.

See also: Weiss, p. 828.

OTHER APPLICATIONS

Games, Economics

★**Davidson, Donald; Suppes, Patrick; and Siegel, Sidney. Decision making: An experimental approach.** Stanford University Press, Stanford, California, 1957. ix+121 pp. $3.25.

In these experiments it is not assumed that subjective probability is the same as objective probability, or that the utility of money is a linear function; and the results seemed to indicate that neither is the case. As a means of testing both, a typical experiment had the following form: At each trial the subject was allowed to select one of two proposed games. One game paid x if a specified chance event occurred, y if it did not; the other paid u or v. By varying the event (e.g., toss of a coin, a die, a pair of dice, etc.) and the payments (in magnitude and sign), estimates were made of subjective probabilities and of utility functions for the various subjects. *A. S. Householder.*

Stelson, Hugh E. Laplace transforms applied to interest functions. Skand. Aktuarietidskr. 39 (1956), 97–104.

Behandlung einiger Beispiele aus der Zinseszinsrechnung mit Hilfe der Laplace-Transformation. Zusammenstellung der entsprechenden Transformationstabellen. Beispiel: Barwert einer nachschüssigen Rente werde mit

$y(t)$ bezeichnet. Dann gilt die Differenzengleichung

$$y(t+1)=vy(t)+v.$$

W. Saxer (Zürich).

Gourary, Mina Haskind. An optimum allowance list model. Naval Res. Logist Quart. 3 (1956), 177–191 (1957).

The author discusses a simplified mathematical model of the allowance list, and draws some general conclusions. *Author's summary.*

Zabel, Edward. Measures of industry capacity. Naval Res. Logist. Quart. 3 (1956), 229–244 (1957).

A survey of statistical estimates of industry capacity is made, and some possible uses for the measures are discussed. *Author's summary.*

Weiss, George H. On the theory of replacement of machinery with a random failure time. Naval Res. Logist. Quart. 3 (1956), 279–293 (1957).

This paper studies the reliability function, and moments thereof, of a machine which is periodically replaced in order to increase the mean time to failure. Two cases are studied: strictly periodic replacement and randomly periodic replacement. *Author's summary.*

See also: Berger, p. 825; Kemeny, p. 860.

Biology and Sociology

Gheorghiu, A.; et Theodorescu, R. Observations sur le calcul de la population urbaine et rurale. Com. Acad. R. P. Romîne 6 (1956), 521–526. (Romanian. Russian and French summaries)

The author connects the functions $R(t)$ and $U(t)$ representing the rural and urban population in a certain area at any moment t by linear equations of Volterra's type (for two populations of different species). These equations are

$$\frac{dR}{dt}=rR-gR,\quad \frac{dU}{dt}=uU+gR,$$

where $r(t)$ and $u(t)$ are specific urban and rural increases, and $g(t)$ the coefficient of migration from the rural to the urban centre. Since $g(t)$ can not be directly determined, it is necessary to introduce an urbanization coefficient $f(t)=U(t)/R(t)$. Then the functions $U(t)$ and $R(t)$ could be determined. The author shows that the results are valid for more restricted zones where each one has only a single urbanization centre, leaving open the question concerning the application of random processes to more general cases. *O. Onicescu* (Bucarest).

See also: Kemeny, p. 860.

Information and Communication Theory

Gel'fand, I. M.; Kolmogorov, A. N.; and Yaglom, A. M. On the general definition of the amount of information. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 745–748. (Russian)

Shannon's definition of "the amount of information contained in the results of experiment A relative to the results of experiment B" is extended to the cases where 1) A and B are subalgebras of a Boolean algebra with a finitely additive probability function, 2) A and B are homomorphisms from a σ-algebra on a given measure space to a σ-algebra with a σ-additive probability function. The usual properties are preserved under this extension. The amount of information is lower semi-continuous in the usual topology. *J. Wolfowitz.*

Syski, R. Analogies between the congestion and communication theories. A. T. E. J. 11 (1955), 220–243.

Congestion theory and communication theory, though having a common object of study and using similar mathematical methods, have influenced each other but little. Following a brief introduction to communication theory, it is pointed out, in particular, that the entropy concept may prove useful in congestion theory. This is illustrated by an example from telephone switching theory. An extensive list of references is provided. *R. Kalaba* (Santa Monica, Calif.).

Pinsker, M. S. Computation of the velocity of communication formation by a stationary random process and the capacity of a stationary channel. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 753–756. (Russian)

A totality of vectors (ξ,η) is called a channel if the conditional probability distribution of η, given ξ, is independent of (ξ,η). The capacity of the channel is the supremum of $J(\xi,\eta)$ where $J(\xi,\eta)$ is Shannon's definition of the amount of information in ξ relative to η. Similar definitions apply to processes $(\xi(t),\eta(t))$. Inequalities are given (without proof), in terms of the determinants of the covariance matrices and of the spectral densities, for the capacities of certain channels not readily describable here. *J. Wolfowitz* (Ithaca, N.Y.).

Rosenbrock, H. H. An approximate method for finding the "best linear servo mechanism". Proc. Inst. Elec. Engrs. C. 103 (1956), 260–266.

Control Systems

Boltyanskiĭ, V. G.; Gamkrelidze, R. V.; and Pontryagin, L. S. On the theory of optimal processes. Dokl. Akad. Nauk SSSR (N.S.) 110 (1956), 7–10. (Russian)

Let the state of a physical system be described at time t by a vector function $x(t)$ with components $x^1,\cdots,x^n$, and let the components of $x(t)$ satisfy a system of differential equations of the form

$$\dot{x}^i=f^i(x^1,\cdots,x^n;u^1,\cdots,u^r)=f(x,u)\quad (i=1,\cdots,n),$$

where $u=(u^1(t),\cdots,u^r(t))$ is a control vector with a prescribed range. The problem considered is that of determining the control vector $u(t)$ in such a way as to minimize the time interval required to change the vector $x(t)$ from a given state ξ_0 to a required state ξ_1 [cf. R. Bellman, I. Glicksberg, and O. Gross, Quart. Appl. Math. **14** (1956), 11–18; MR **17**, 1206]. In the present paper, necessary conditions for a minimum are derived under suitable smoothness assumptions, as are sufficient conditions for a local minimum. *E. F. Beckenbach.*

Rumyancev, V. V. On the theory of stability of regulated systems. Prikl. Mat. Meh. 20 (1956), 714–722. (Russian)

The control systems studied here are characterized by

the equations

$$(1)\quad \eta_i'=\sum_{\alpha=1}^{n} b_{i\alpha}\eta_\alpha+\sum_{\beta=1}^{k} h_{i\beta}f_\beta(\sigma_\beta)\ (i=1,\cdots,n)$$
$$\sigma_\beta=\sum_{i=1}^{n} j_{\beta i}\eta_i\ (\beta=1,\cdots,k),$$

where the η_i are the controlled coordinates, the σ_β are the coordinates of the regulating organs, the $f_\beta(\sigma_\beta)$ represent the characteristics of the servomotors, and the $b_{i\alpha}$, $h_{\beta i}$, $j_{\beta i}$ are constant parameters. Using the techniques of Lur'e [Some nonlinear problems of the theory of automatic regulation, Gostehizdat, Moscow-Leningrad, 1951; MR 15, 707], equations (1) are reduced to the canonical forms

$$(2)\quad x_i'=\sum_{j=1}^{n} a_{ij}x_j+\sum_{s=n-k+1}^{n} g_{is}f_s(x_s)\ (i=1,\cdots,n),$$

and

$$(3)\quad z_i'=\lambda_i z_i+\sum_{s=1}^{k} H_{is}f_s(\sigma_s)\ (i=1,\cdots,n),$$
$$\sigma_\beta=\sum_{\alpha=1}^{n} \gamma_{\beta\alpha}z_\alpha\ (\beta=1,\cdots,k),$$

where the a_{ij}, g_{is}, H_{is} and $\gamma_{\beta\alpha}$ are constants, and the λ_i are zeros of $|b_{i\alpha}-\lambda\delta_{i\alpha}|=0$. Assuming that the $f_s(x_s)$ can be expressed as $f_s(x_s)=[c_s+\varepsilon\varphi_s(x_s)]x_s$ $(s=n-k+1,\cdots,n)$, where ε is a parameter, the c_s are constants independent of ε, and the $\varphi_s(x_s)$ are bounded functions of x_s, the author constructs a Lyapunof function for (2) by an extension of a method employed by Letov for the case $k=2$ [Prikl. Mat. Meh. 17 (1953), 401–410; MR 15, 707], and obtains a set of sufficient conditions for the asymptotic stability of (1) in terms of ε and the $\varphi_s(x_s)$. Similar results are found for (3) under the additional assumption that $\sigma_s f_s(\sigma_s)>0$ $(s=1,\cdots,k)$. *L. A. Zadeh* (New York, N.Y.).

Stout, T. M. Basic methods for nonlinear control-system analysis. Trans. A. S. M. E. **79** (1957), 497–507, discussion 507–508.

Nikiforuk, P. N.; and West, J. C. The describing-function analysis of a non-linear servo mechanism subjected to stochastic signals and noise. Proc. Inst. Elec. Engrs. C. **104** (1957), 193–203.

A technique is presented for evaluating the response of a specific type of non-linear servo-mechanism to random signals, to sinusoidal signals contaminated by noise and to random signals contaminated by noise. In particular, a second-order servo-mechanism incorporating derivative-of-error stabilization and subjected to torque limination is considered. Theoretical responses are evaluated for a wide range of parameters, and experimental verification is included. *From the authors' summary.*

Povarov, G. N. On the study of symmetric Boolean functions from the point of view of the theory of relay-contact circuits. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 183–185. (Russian)

A simple geometric criterion based on a representation of Boolean functions is derived for determining when a Boolean function is symmetric, and applications of this theorem are given, including the case of circuits with independent contacts. *C. Saltzer* (Syracuse, N.Y.).

Trahtenbrot, B. A. Synthesis of nonrepeating circuits. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 973–976. (Russian)

A non-repeating circuit is a circuit in which distinct branches are associated with distinct Boolean variables. By graph-theoretical methods it is shown that if a Boolean function is realizable by a non-decomposable circuit then the circuit is unique up to an isomorphism. If the Boolean function is realizable by a decomposable circuit then any realization can be transformed into any other realization by a finite number of reflections of its sub-circuits and an isomorphic mapping. A method for the synthesis of these circuits is given. *C. Saltzer* (Syracuse, N.Y.).

See also: Jarre, p. 834.

HISTORY, BIOGRAPHY

Schröder, K. Riemanns Habilitationsvortrag und seine Auswirkungen in Mathematik und Physik — ein historischer Überblick. Schr. Forschungsinst. Math. **1** (1957), 14–26.

Jarník, Vojtěch. Ten years of mathematics in liberated Czechoslovakia. Časopis Pěst. Mat. **80** (1955), 261–273. (Czech)

Lampariello, G. B. Riemanns physikalisches Denken. Schr. Forschungsinst. Math. **1** (1957), 222–234.

★ **Крылов, А. Н. [Krylov, A. N.] Воспоминания и очерки. [Recollections and essays.]** Izdat. Akad. Nauk SSSR, Moscow, 1956. 884 pp. 30 rubles.

MISCELLANEOUS

★ **Kemeny, John G.; Snell, J. Laurie; and Thompson, Gerald L. Introduction to finite mathematics.** Prentice-Hall, Inc., Englewood Cliffs, N. J., 1957. xi+372 pp. $5.00.

On the inside of the cover of this book the publishers remark "along with the originality of approach, considerable material not previously available in elementary form is included in 'Introduction to finite mathematics'". Here is a rare instance in which a publisher is guilty of understatement. This book is more than original. It is unique.

In recent years there has been increased public awareness of the important role played by mathematics in many aspects of contemporary life. This has brought about a demand at many institutions for "terminal" courses in mathematics of one or at most two semesters suitable for students who may not intend to go further in the subject as well as potential concentrators. Such a course would attempt in this brief period to indicate what mathematics is and what it can do. The usual analytic geometry, calculus is in some ways unsuitable for this purpose, as such courses are generally designed as part of a larger program which also includes related material in the physical sciences. The book under review takes an entirely new tack. Instead of elementary analysis the mathematical subject matter is "finite mathematics" (actually certain branches of algebra). Instead of physical applications the book draws its illustrative material from the biological and social sciences.

In the authors' own words the book's purpose is "to introduce a student to some concepts in modern mathematics early in his college career. While primarily a mathematics course, it was to include applications to the biological and social sciences and thus provide a point of view, other than that given by physics, concerning the possible uses of mathematics. Our aim was to choose topics which were initially close to the student's experience, which are important in modern day mathematics, and which have interesting and important applications."

As to the somewhat mystifying title of the book, the authors explain "our purpose... was to develop several topics from a central point of view. In order to accomplish this on an elementary level, we restricted ourselves to consideration of finite problems, that is, problems which do not involve infinite sets, limiting processes, continuity, etc."

The program of the book in very brief outline, is the following: The first chapter, "Compound Sentences", deals with the elementary propositional calculus, the central idea being that of truth tables. Also in this chapter the notion of the "logical possibilities" associated with a statement is introduced. This concept, which appears to be a device invented by the authors (we return to this later), provides a link between the first and second chapter ("Sets and Subsets"); the latter's main concern is the Boolean algebra of sets and its relation to the propositional calculus of the previous chapter. The set notion leads naturally to Chapter III, "Partitions and Counting", which is concerned with elementary combinatorics, permutations, etc. The material of all the preceding chapters is brought together in Chapter IV, "Probability Theory", which, however, is quite a departure from the usual treatments of this subject in first year texts as indicated by the presence of sections entitled "Finite Stochastic Processes", "The Law of Large Numbers", "Markov Chains". This last section leads to Chapter V on "Vectors and Matrices" which, after introducing the standard terms and operations, applies them to the theory of Markov chains. These five chapters comprise the "basic core" of the book and represent material which could be conveniently covered in one college semester. The book concludes with two "optional" chapters (so indicated by an asterisk). The first entitled "Linear Programming and the Theory of Games", contains an elementary treatment of some aspects of these two subjects. The final chapter "Applications to Behavioral Science Problems" treats five problems, "selected for their interest both to mathematicians and to behavioral scientists. One topic was chosen from each of five sciences: sociology, genetics, psychology, anthropology, and economics."

This book represents such a radical departure from traditional college texts that it would take considerable space to give it a really adequate review. Perhaps some notion of its unusual flavor can be conveyed by listing a few of the many (over 1000) varied and excellent problems which appear at the ends of each section.

In a starred (optional) section of Chapter I on "Statements Having Given Truth Tables" the student is led in a series of exercises to the proof that every logical connective can be expressed in terms of "neither...nor". In a section on "Valid Arguments" the authors develop the theory of the syllogism, giving the student a systematic procedure for checking the validity of arguments like "Father praises me only if I can be proud of myself. Either I do well in sports or I cannot be proud of myself. If I study hard then I cannot do well in sports. Therefore, if Father praises me then I do not study hard."

The final section (starred) of this chapter is on "Application to Switching Circuits". Problem: "A committee has five members. It takes a majority vote to carry a measure except that the chairman has a veto. Design a circuit for the committee, so that each member votes for a measure by pressing a button, and a light goes on if and only if the measure is carried."

In the chapter on sets the following problem is typical. "In a survey of 100 students the numbers studying various languages were found to be: Spanish, 28; German, 30; French, 42; German and French, 5; all three languages, 3. (a) How many students were studying no language? (b) How many students had French as their only language?" etc. In the chapter on partitions we find, "On a transcontinental airliner there are 9 boys, 5 American children, 9 men, 7 foreign boys, 14 Americans, 6 American males and 7 foreign females. What is the number of people on the plane?"

As a final example we quote one of the problems on Markov chains. "A professor tries not to be late for class too often. If he is late one day, he is 90% sure to be on time the next. If he is on time, then the next day there is a 30% change of his being late. In the long run how often is he late?"

The foregoing examples show the way in which each new mathematical concept is illustrated by a non-trivial application. Even such seemingly mechanical matters as the rules for matrix multiplication are illustrated by realistic examples in which multiplying matrices is precisely the natural thing to do. Indeed, of the text's many virtues, perhaps the outstanding one is the remarkable balance which the authors have preserved between abstract theory and "common sense" applications. This is "applied mathematics" in the best sense.

A second notable feature of the book is its emphasis on pedagogical devices of a graphical nature. Simple problems in set theory are solved by the use of Venn diagrams. Another device which occurs in the first chapter and is used with especial effectiveness in the probability chapter is that of "tree diagrams" to represent either partitions or logical possibilities at various stages of some process. This use of "picture writing" is thoroughly exploited to the great advantage of the student.

Having pointed out some of the book's strong points, we now turn to what, in the reviewer's opinion, is a very serious flaw. The authors have chosen as their central notion the concept of a "statement". Almost all other notions in the book are derived from or closely related to this one. In the chapter on probability the fundamental notion is not the usual idea of the probability of an event but the "probability of a statement". It is therefore somewhat of a shock to find that this notion is used in a completely inconsistent manner. Specifically, the book opens with the following sentences. "A statement is a verbal or written assertion. In the English language such assertions are made by means of declarative sentences." But on page 21 we find, "Hence, also, the statement 'Draw three white balls' is logically false." (Problem for the authors: what is a declarative sentence?) Next, on page 2 we find, "The fundamental property of any statement is that it is either true or false (and that it cannot be both true and false);" but on page 19 we read, "Normally, for a given statement there will be many cases in which it is true and many in which it is false." Which

sentence is the student to believe? Worse, what is the instructor to tell him?

The last example cited above does not represent a momentary lapse on the part of the authors, but a deliberate ambiguity which recurs throughout the book. They evidently did not want to get involved in the distinction between sentences like "$x^2=4$" and "p implies q" which are not statements (they are "propositional functions") and are true in some cases, false in others, as against honest to goodness statements like "Socrates is a man" and "2+2=5", which are simply true or false — no cases about it. Whatever the authors' intention may have been, the result is to leave the instructor in an unhappy dilemma of having either to go along with the deception, or rewrite a substantial portion of the exposition himself. He will be especially uncomfortable if he is pressed to define what is meant by the "logical possibilities of a statement". This very central notion of the book is, in fact, nowhere defined, because, indeed, no definition is possible; yet many of the other key concepts, probability included, are defined in terms of it. Most people would agree that an elementary text is not the place to be stuffy on matters of rigor, but it is also not the place to perpetrate obvious inconsistencies, for you can't fool all of the students all of the time, and it is precisely the ones that can't be fooled who should be the especial concern of text-book writers.

The book has other shortcomings in the small. Words like "coordinate" (of a point) and "transformation" suddenly appear without having ever been defined. Some of the exposition, especially the description of the Ester learning model in the final chapter, is rather muddy. Also the authors are not quite faithful to their vows of finiteness. They make free use of the sets of real and natural numbers, without even apologizing, and we also read that if "P is a regular stochastic matrix ... then if p is a probability vector, pP^n approaches" — "But Sir, what does 'approaches' mean?" In a previous paragraph we were told parenthetically that "approaches" means "gets close to". Just to keep the record straight, the authors might have noted that this idea of "closeness" is in fact made precise in books on "infinite mathematics". These are small matters, however, which the instructor can easily remedy.

The last two chapters of the book should really be considered as a separate unit since the level of difficulty, especially in the final chapter, is much higher than in the first five. As already noted, the exposition here seemed weak in spots. The examples treated were, however, uniformly interesting and, I suspect, entirely unfamiliar to nine out of ten teachers of college mathematics.

Finally, this book should be especially appreciated by specialists in fields other than mathematics, particularly behavioral scientists, for it makes available for the first time the sort of elementary mathematical training appropriate for students whose main interest lies in this field.

D. Gale (Providence, R.I.).

Menger, Karl. What are x and y? Math. Gaz. **40** (1956), 246–255.

The author argues that radical ambiguities in the terms 'x' and 'y' make the teaching of algebra, analytic geometry, and analysis unnecessarily difficult and their presentation as "a system of formulae connected by articulate rules" virtually impossible (p. 255). Reviving Newton's term "fluent", defined as a "consistent class of quantities" — where "quantity" in turn is defined as an ordered pair consisting of a number and "anything else" —, he distinguishes sharply between fluents and variables (the latter being characterized as "replaceable symbols"). Only those fluents whose domains are numbers or systems of numbers should properly be called "functions."

On the basis of this distinction, Menger goes on to list, with copious and varied examples, twelve disparate uses of the "duodeciguous" symbols 'x' and 'y' as: (1) numerical variable, (2) identity function, (3) selector function, (4) real-valued complex function, (5) indeterminate, (6) part of operational symbol, (7) operator, (8) specific fluent (abscissa, ordinate), (9) function variable, (10) fluent variable, (11) sundry uses (some parallel to random variable), (12) dummy (no assignable meaning). Classical treatises on analysis, he claims, fail to distinguish clearly between uses (1) "$D \sin x=\cos x$ for any x," (2) "$d \sin x/dx=\cos x$," and (10) "$d \sin x/dx=\cos x$ for any fluent x."

The author proposes notation molded to reflect these multiple uses: e.g., roman (instead of italic) type for (1); the letters 'f, g, h' for (9) and 'u, v, w' for (10); 're' and 'im' for (4); even certain typographical symbols — '$*$' and '†'. In many cases, he feels, the result would not only be free from present ambiguities, but would also be closer to the way mathematicians talk, as opposed to the way they often write.

Three comments seem in order: I. 'x' and 'y' are certainly used ambiguously in many textbooks and "classical treatises"; the author has rendered a service by underlining this fact. But does the same situation exist in current technical papers and monographs? Are not the ambiguities at this level both less ubiquitous and less alarming than the author suggests?

II. Even where specifiable ambiguities of notation exist, do they in fact seduce practicing mathematicians or students of mathematics into confusion of distinct concepts or operations? The author himself admits that the ambiguity of 't' in physics (standing for both time and temperature) has never caused a physicist to confuse the two quantities. But he insists that notational ambiguities in mathematics are "more subtle and insidious" (p. 253). Is this really the case?

III. Granting that at least some of the ambiguities exposed by the author might profitably be eliminated, is there not the danger that the cure (new and unfamiliar complexities of notation) might, in the short run, be worse than the disease, actually generating a new layer of conceptual confusion?

G. L. Kline.